Non-Hermitian quantum field theory

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King's College London

Non-Hermitian Quantum Field Theory

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A dissertation submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

of

King's College London.

Department of Physics

King's College London

June 3, 2020

I, Dries Seynaeve, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

The main objective of this thesis is to provide a better understanding of non-Hermitian, $\mathscr{PT}$-symmetric Quantum Field Theories. In particular we focus on a non-Hermitian extension of the scalar sector of the Standard Model.

Firstly, we consider a non-Hermitian Lagrangian that consists out of two complex scalar fields. We discuss a consistent manner to define the equations of motion and we reexamine the relation between transformations and conserved currents. Because of the non-Hermitian behaviour of our system the relation between conserved currents and symmetries, known as the Noether's theorem, no longer holds. We later discuss spontaneous symmetry breaking of our scalar model and show that the Goldstone theorem still applies for our non-Hermitian scalar model. We show that the Goldstone theorem relies on the existence of a conserved current, whose transformation breaks the vacuum. As discussed before, this transformation will not be a symmetry of our system. Additionally, we show how the conventional quantisation of the path integral formulation should be extended consistently for $\mathscr{PT}$-symmetric, non-Hermitian systems.

Secondly, we include an Abelian gauge field into our theory. Ensuring Gauge invariance is nontrivial for this model. We dicuss the problems that occur and propose a method to build a consistent theory. We then discuss a non-Hermitian extension to the Englerd-Brout-Higgs mechanism for mass generating of the gauge field. Finally, we also include non-Abelian $SU(2)$ gauge fields and naturally end up with a non-Hermitian two-Higgs-doublet model extension to the Standard Model. We then compare its mass spectrum to that of a Hermitian two-Higgs-doublet model.

Acknowledgements

I first would like to thank my supervisor Jean Alexandre for the help and support he has given me over the past years. The vision and guidance he has provided me with were invaluable in completing this work.

I would also like to thank Peter Millington. His thoroughness and insight have helped me understand and appreciate the complexities of this work.

I am also very grateful to John Ellis. His knowledge and expertise were a great help and provided fresh perspectives whenever I got stuck.

I would like to express my gratitude to the entire Department of Physics, especially to James Alvey, Tarso Franarin, David Rodriguez, Sreedevi Varma, Kareem Farrag, Mohamed Ali Al-Badri and Nashwan Sabti.

I would also like to thank my friends and family. It is because of their continuous support that all of this has been possible.

I am especially grateful to Lien and Maurice, who both have always supported me.

Lastly, I would like to thank my friends Imran and Pauline for always being there for me.

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Chapter 1

Introduction

One of the most tested and studied theories in particle physics is the Standard Model of physics. This model has had many successes since its conception. The most famous being the prediction of the Higgs particle which has been experimentally verified in 2012. Other achievements include the prediction of the $W^{\pm}$ and Z bosons, their masses and also the anomalous magnetic dipole moment of the electron. Despite these remarkable predictions, we know that the Standard Model can not be our final theory of nature. It fails at providing a consistent description of gravity or explaining neutrino oscillation. Because of this, many people have been looking at different possible extensions to the Standard Model. One such possible extension is the multi-Higgs doublet model. In this work, we construct and study a non-Hermitian two-Higgs-doublet model.

As is the case for most theories, the Standard Model does assume Hermicity for all its fundamental operators. Recently, however, there has been a growing interest in Quantum models that possesses non-Hermitian Hamiltonians but are instead $\mathscr{P}\mathscr{T}$-symmetric. In works such as [2–7], such models have been shown to possess real energies that are bounded below as long as they possess an unbroken $\mathscr{P}\mathscr{T}$-symmetry. Such models can be seen as non-trivial extensions of traditional Quantum Mechanics into the complex plane [2, 8, 9]. These theories might seem only useful as a mathematical curiosity but have led to some interesting experimental results. This area of physics has proven to be very fruitful in experiments concerning optics as discussed in [10–15]. Other models have also led to interesting research in the field of photonics [16–20], superconducting wires [21, 22] and $\mathscr{PT}$-symmetric electronic circuits [23]. Most of these experiments exploit the behaviour of coupled gain and losses, which is present in all unbroken $\mathscr{PT}$-symmetric theories.

There have also been extensive studies done into non-Hermitian Quantum Mechanical systems [24–32]. The work [33] describes the possibility of a smooth transition between Hermitian and $\mathscr{PT}$-symmetric phases in Quantum Mechanics. The non-trivial relation between symmetries and conserved currents are commented on in works such as [34] and the Von Neumann entropy for non-Hermitian Quantum systems is discussed in [35].

For non-Hermitian Quantum Field Theories there have been studies done for a model with imaginary interaction potentials [36], including models with an $i\phi^3$ scalar interaction [37–42]. It was shown in [43] that this still is a meaningful effective potential despite being unbounded from below. This was also shown for a $\mathscr{PT}$-symmetric model with $-\phi^4$ interaction, featuring asymptotic freedom [44]. A $\mathscr{PT}$-symmetric fermionic model, with non-Hermitian mass term $\mu\overline{\psi}\gamma^5\psi$ was considered in [45]. This model has been studied further in [46], where it is shown that a conserved current can be defined, with a corresponding density of probability for left- and right-handed components depending on the ratio μ/m. In the critical limit, one of these two components disappears from the spectrum, and the non-Hermitian features of the model allow us to suppress one chirality continuously. A similar result is found in [47], where a non-Hermitian lattice fermionic system exhibits non-equal numbers of right- and left-handed fermions. This non-Hermitian model can be used to provide an alternative description to neutrino masses [48–50] or dark matter [51]. Non-Hermitian extensions to Quantum Field Theory have also been applied to describe neutrino oscillation [52] and the decay of the Higgs boson [53]. Effective non-Hermitian Hamiltonians with complex spectra are also known to play a role in the description of unstable systems with particle mixing [54].

In this work, we study a non-Hermitian scalar Quantum Field Theory. This might provide a non-Hermitian extension to the Higgs sector in the Standard Model. The outline of this work is as follows. In Chapter 2, we discuss the role of symmetries in Quantum Systems. We discuss both continuous and discrete transformations and give examples of them. We show how the connection between symmetries and conserved currents, known as the Noether's theorem [55], hinges on the Hermicity of our system. We will discuss later how to deal with this for our non-Hermitian scalar model. Afterwards, we focus our attention onto a review of non-Hermitian systems. We show that those systems with an unbroken $\mathscr{PT}$-symmetry can be physically consistent. These theories have real energies and, given a suitable inner product, they can have unitary time evolution. Another remarkable feature of such non-Hermitian models that we will study is the relationship between the left and right eigenfunctions. We then conclude this chapter with a review of a family of non-Hermitian Quantum Mechanical models with potential given by $V(x) = \hat{x}^2 (i\hat{x})$. This family of models has been extensively studied in works such as [1, 2, 4–6, 24].

In Chapter 3, we introduce a non-Hermitian scalar Quantum Field Theory consisting of two complex scalar fields. For this Lagrangian, we discuss the eigenmasses and the conditions on the couplings that assure the reality of them. We also derive the discrete $\mathscr{P}$- and $\mathscr{T}$-transformations for this system. The equations of motion for this system are not trivial since the standard Euler-Lagrange equations do not possess dynamical solutions. Instead of the normal variation of the action with respect to both the fields and their complex conjugates, we should choose only one of these. It turns out that this choice in equations of motion does not change the physical observables. However, this deviation from Hermitian theories will be significant when we revisit the Noether's theorem. Because the variation with respect to the fields and the complex conjugate fields cannot both simultaneous be zero, the normal Noether's current of a symmetry will in general not be conserved. We discuss the condition a transformation should meet to have a corresponding conserved current. For the second part of this chapter, we review the results we derived for the scalar model and apply them to the Fermionic model that was discussed in [45, 46, 49].

In Chapter 4 we discuss the Englert-Brout-Higgs mechanism [56, 57] for mass generation of the $U(1)$ gauge field. We start this study by discussing the spontaneous symmetry breaking and the Goldstone theorem. We formally show that the existence of the Nambu-Goldstone mode requires the existence of a conserved current whose transformation will not leave the non-trivial vacuum invariant. As we discussed before, in non-Hermitian systems, such a transformation will not be a symmetry. We then calculate the eigenmasses of this system together with their corresponding left and right eigenmodes. After discussing the Goldstone theorem, we introduce a $U(1)$ gauge field into our theory and examine how to consistently describe gauge invariance for our system. To have a transverse polarisation tensor, it turns out that the gauge field should couple to a non-conserved current. As a consequence of this, one needs to introduce gauge fixing terms in our Lagrangian to obtain consistent Maxwell equations. We are then able to discuss the Englert-Brout-Higgs mechanism and show that this mechanism still works in a non-Hermitian setting. Finally, we compare the work that was done in this chapter with a different interpretation formulated by Mannheim [58]. The author suggests an alternative approach to the equations of motion using a similarity transformation instead. With this alternative formulation, the author shows that the Englert-Brout-Higgs formulation still holds but leads to a different mass for the gauge field.

Next, we are able to discuss path integral quantisation in Chapter 5 for non-Hermitian, but unbroken $\mathscr{P}\mathscr{T}$-symmetric models. As discussed in Chapter 2, our system possesses a $\mathscr{C}'\mathscr{P}\mathscr{T}$-symmetry. This should be reflected in the definition of our partition function. We show that the partition function should be defined as a path integral of $\mathscr{C}'\mathscr{P}\mathscr{T}$-conjugate fields and that the source terms should also be related by a $\mathscr{C}'\mathscr{P}\mathscr{T}$-transformation. The consistent description of the partition function ensures us to do calculations beyond the classical level. We do this by calculating the one-loop 1PI effective action for our scalar Quantum Field Theory. For this system, we are then able to show the running of our couplings. From there, it can be shown that we have a Hermitian fixed point, given that the only non-Hermitian terms stem from the mass matrix. In Chapter 5, we also show the consistency of the Goldstone theorem beyond the classical level and calculate the Goldstone mode at the one-loop level. Lastly, we show the consistency of our model, proposed in Chapter 4, beyond the one-loop level, by showing the reality of the $U(1)$ gauge field.

Subsequently, we expand upon our model by including a $SU(2)$ gauge field. With this in mind, we upgrade the scalar fields to scalar doublets and end up with a non-Hermitian two-Higgs-doublet model. We start Chapter 6 by reviewing the BRST transformations and the gauge fixing procedure for non-Abelian gauge fields. Afterwards, we discuss a Hermitian two-Higgs-doublet model that has been studied in works such as [59–62]. This enables us to focus our attention towards our non-Hermitian two-Higgs-doublet model. We first study the scalar sector of this model and discuss the spectrum before symmetry breaking. Afterwards, we can gauge this model so that it is invariant under a $U(1) \times SU(2)$ transformation. As was the case for the Abelian gauge field, we need to include gauge fixing terms into the Lagrangian to have consistent equations of motion. This leads to a gauge restriction of the model that we find explicitly by making use of the BRST symmetry of our system. This allows us to calculate the scalar spectrum after spontaneous symmetry breaking around the non-trivial vacuum.

Once we have derived the physical spectrum of our theory it is possible to set conditions on our parameters to allow or system to be physical. Once these conditions have been derived, we can discuss the critical limits of our theory. Lastly, for this chapter, we can compare the Hermitian two-Higgs-doublet model we discussed before with our non-Hermitian two-Higgs-doublet model. Finally, in Chapter 7, we present our concluding remarks and discuss possible directions for future work.

Chapter 2

Symmetries and Unbroken $\mathscr{P}\mathscr{T}$-symmetric Systems

An interesting method to study physical systems is to examine how such a system changes under different transformations. This can provide us with a better understanding of such systems and their physical implications. For example, a time-reversal symmetry of a system $H(t)$,

$$H(t) = H(-t) \ , \tag{2.1}$$

implies that such a system $H(t)$ can not see the arrow of time. By this, it is meant that one can not physically distinguish between processes that occur forward or backwards in time. As another example, we consider a system that describes the movement of two particles

$$H(r_i, v_i) \quad \text{where} \ \ i \in \{1,2\} \ , \tag{2.2}$$

where r_i, v_i are the position and velocity respectively of a particle i. A symmetry of the form $v_1 \leftrightarrow v_2$, $r_1 \leftrightarrow r_2$ implies that one can not physically distinguish between these two particles.

From these examples, it is clear that the presence of certain symmetries implies physical properties for our system. Alternatively, when building physical models like the Standard Model, one should impose certain symmetries because of the known physical constraints they entail. For example, if a physical process is as likely to occur forwards as backwards in time, the mathematical model that should describe this system should reflect this.

Things are slightly different for Hermitian systems. A system is Hermitian if

$$H = H^{\dagger} \,, \tag{2.3}$$

where the $\dagger$-operator is the transpose operator combined with complex conjugation. This symmetry is normally imposed to ensure real energies and unitary time evolution. Reversely, however, it is not the case that unitary time evolution and real energies can only occur when the system is Hermitian. In fact, it turns out that having a different symmetry might also give this. This symmetry is known as unbroken $\mathscr{P}\mathscr{T}$-symmetry.

A transformation that consists of complex conjugating seems to be mathematical in nature. There is no clear or straightforward interpretation as to how this changes the state physically. Such an interpretation does however exist for $\mathscr{P}\mathscr{T}$-conjugation, since parity and time are physical properties. This makes studying such systems particularly interesting.

We start this section by discussing continuous and discrete symmetries. Later on, we discuss how such unbroken $\mathscr{P}\mathscr{T}$-symmetric models should be dealt with. We will show that the spectrum of such models is real. These systems will also have a unitary time evolution with respect to a $\mathscr{C}'\mathscr{P}\mathscr{T}$-inner product. Lastly, we end this chapter by discussing a particular family of unbroken $\mathscr{P}\mathscr{T}$-symmetric Quantum Mechanical systems. We show the classical paths for these systems.

2.1 Continuous and Discrete Symmetries

2.1.1 Continuous Transformations

Continuous transformations play an important role in physics. A continuous transformation is a transformation φ_t that describes a continuous change of a system with respect to t. It is defined such that for $t = 0$, the transformation is equal to the identity transformation

$$\varphi_0 = id \ . \tag{2.4}$$

Consequently, for t infinitely close to 0 one can linearize the transformation as

$$\varphi_t = \mathbb{I} - itA + \mathscr{O}\left(t^2\right) \ , \tag{2.5}$$

where the components of A are the generators of this continuous transformation. We will focus our attention on this region in parameter space where t is infinitely close to 0. In this infinitesimal region, these transformations form a group. A primary example of this would be the unitary group of global $U(1)$ transformations acting on the space $\mathbb{R}^n$ and of the form

$$\varphi_t^{U(1)} = e^{it} = \mathbb{I}_n - it\mathbb{I}_n + \mathscr{O}(t^2) \ . \tag{2.6}$$

The generator of this transformation is the identity matrix $\mathbb{I}$. This group is particular in that the order in which the transformations are applied does not affect the result. Such groups are known as Abelian groups and they will in general have commuting generators $\{K_1, K_2, \cdots, K_N\}$.

$$[K_i, K_j] = 0 \ , \qquad \forall i, j \in \{1, 2, \ldots, N\} \ . \tag{2.7}$$

Another example of a group of transformation would be the Lie group consisting of the proper Lorenz transformations. These are defined as the transformations $\Lambda \in \mathbb{R}^4$ that act on space-time vectors $x^\alpha = (t, x, y, z)^T$ such that $s^2 = x^\alpha g_{\alpha\beta} x^\beta$ is invariant and with positive determinant.

$$s^2 = x^\alpha g_{\alpha\beta} x^\beta = x^\alpha \Lambda^T g_{\alpha\beta} \Lambda x^\beta \ \ , \ \ |\Lambda| = +1 \ , \tag{2.8}$$

where $g_{\alpha\beta}$ is the Minkowski metric. These transformations can be writen as

$$\Lambda = e^{iw^{\alpha\beta}M_{\alpha\beta}} \quad , \quad w^{\alpha\beta} \in \mathbb{R} \, , \tag{2.9}$$

where $w^{\alpha\beta} = -w^{\beta\alpha}$ and $M^{\alpha\beta} = -M^{\beta\alpha}$. The tensor $w^{\alpha\beta}$ consist out of the 6 parameters of the transformation and $M^{\alpha\beta}$ are the 6 generators of the Lorenz transformations, given by

$$M_{01} = K_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \; M_{23} = S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \tag{2.10}$$
$$M_{02} = K_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \; M_{31} = S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$
$$M_{03} = K_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} , \; M_{12} = S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

The S_i matrices are the generators responsible for rotations and the K_i matrices generate boosts. The commutation relations between these generators is given by the expression

$$[M_{\alpha\beta}, M_{\gamma\delta}] = i\left(-g_{\alpha\gamma}M_{\beta\delta} + g_{\beta\gamma}M_{\alpha\delta} - g_{\alpha\delta}M_{\gamma\beta} + g_{\beta\delta}M_{\gamma\alpha}\right) \, . \tag{2.11}$$

Remark that these commutators are in general not zero, and thus the proper Lorenz transformations form a non-Abelian group.

2.1.2 Continuous Symmetries

If a continuous transformation leaves a physical system invariant, this transformation is a continuous symmetry. Let us consider a function $f(x^\alpha)$, depending on the space-time position x^α. If this function is invariant under the proper Lorenz transformations

$$f(x) = f(\Lambda x) \ , \tag{2.12}$$

then the Lorenz transformations are symmetries of the function $f(x^\alpha)$. This concept can be generalized when we consider physical quantities depending on fields ϕ_i, which are in turn themselves dependent on the spacetime coordinates x^α. In that case, we will need to know how both the coordinates x^α and the fields ϕ_i transform under transformations. For example, the action of a Quantum Field Theory $S[\phi(x)]$ transforms as

$$S[\phi(x)] \to S'[\phi(x)] = S[\phi'(x')] \ . \tag{2.13}$$

Continuous symmetries play an essential role in physics since in Hermitian systems there is a known one-to-one correspondance between a continuous symmetry and a conserved current. This theorem, which was proven in 1918 by Noether [55] is known as the Noether theorem. The proof of this theorem is straightforward but does make use of the Euler-Lagrange equations.

Proof. Let φ_t be a continuous infinitesimal symmetry of the Lagrangian. Under this transformation the fields ϕ_i transform as

$$\varphi_t : \phi_i(x^\alpha) \to \phi_i'(x'^\alpha) \ . \tag{2.14}$$

We use the notation

$$\delta\phi_i(x^\alpha) \equiv \phi_i'(x^\alpha) - \phi_i(x^\alpha) \tag{2.15}$$

$$\delta_T\phi_i(x^\alpha) \equiv \phi_i'(x'^\alpha) - \phi_i(x^\alpha) \ . \tag{2.16}$$

The Lagrangian $\mathscr{L}$ is a functional depending on the fields $\phi_i(x^\alpha)$ and $\partial_\alpha\phi_i(x^\alpha)$. The fact that φ_t is a continuous symmetry of the Lagrangian is expressed as

$$\delta_T \mathscr{L} = \mathscr{L} - \mathscr{L}' = 0 \ . \tag{2.17}$$

The quantities $\delta_T \phi_i$ and $\delta \phi_i$ are related as

$$\begin{aligned}\delta_T \phi_i &= \phi_i'(x'^\alpha) - \phi_i(x^\alpha) = \phi_i'(x'^\alpha) - \phi_i(x'^\alpha) + \phi_i(x'^\alpha) - \phi_i(x^\alpha) \\ &= \delta \phi_i + \partial_\alpha \phi_i \delta x^\alpha \ ,\end{aligned} \tag{2.18}$$

since x^α and x'^α are infinitely close. We can now express equation (2.17) as

$$\begin{aligned}0 &= \delta_T \mathscr{L} = \mathscr{L}(\phi_i + \delta_T \phi_i, \partial_\alpha \phi_i + \delta_T(\partial_\alpha \phi_i)) - \mathscr{L}(\phi_i, \partial_\alpha \phi_i) \\ &= \mathscr{L}(\phi_i, \partial_\alpha \phi_i) + \frac{\partial \mathscr{L}}{\partial \phi_i} \delta_T \phi_i + \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi_i)} \partial_\alpha \delta_T \phi_i - \mathscr{L}(\phi_i, \partial_\alpha \phi_i) \\ &= \frac{\partial \mathscr{L}}{\partial \phi_i} \delta \phi_i + \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi_i)} \partial_\alpha \delta \phi_i + \frac{\partial \mathscr{L}}{\partial \phi_i} \partial_\alpha \phi_i \delta x^\alpha + \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi_i)} \partial_\alpha \left(\partial_\beta \phi_i \delta x^\beta \right) \\ &= \partial_\alpha \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi_i)} \delta \phi_i + \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi_i)} \partial_\alpha \delta \phi_i + \frac{\partial \mathscr{L}}{\partial \phi_i} \partial_\alpha \phi_i \delta x^\alpha + \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi_i)} \left(\partial_\alpha \partial_\beta \phi_i \right) \delta x^\beta \\ &= \partial_\alpha \left(\frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi_i)} \delta \phi_i \right) + \partial_\alpha \mathscr{L} \delta x^\alpha = \partial_\alpha \left(\frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi_i)} \delta \phi_i + \mathscr{L} \delta x^\alpha \right) \ .\end{aligned} \tag{2.19}$$

From this it follows directly that $j^\alpha = \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi_i)} \delta \phi_i + \mathscr{L} \delta x^\alpha$ is indeed a conserved current since

$$\partial_\alpha j^\alpha = 0 \ . \tag{2.20}$$

□

Remark that in the fourth line of equation (2.19) we explicitly used the Euler Lagrange equations

$$\partial_\alpha \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi_i)} - \frac{\partial \mathscr{L}}{\partial \phi_i} = 0 \ . \tag{2.21}$$

We will show later that these equations are no longer all valid when we consider non-Hermitian Lagrangians. The Noether theorem, as stated here, will thus no longer hold and we will need to be more careful in finding a correspondance between conserved currents and transformations.

2.1.3 Discrete Symmetries

By definition a continuous transformation φ_t can always be changed in a continuous manner until we obtain the identity transformation $\varphi_0 = id$. These transformations are used to describe a continuous perturbation of our system. Remark for example how the proper Lorenz transformations described a continuous change in the space-time coördinates $\mathbf{x} = (t,x,y,z)^T$. Discrete transformations, as their name suggests, are instead used to describe a discrete change in our system.

We discuss two important discrete transformations in this section. Firstly, the time reflection opperation $\mathscr{T}$ that acts on the space-time coördinates by changing the sign of the time coördinate

$$\mathscr{T} : (t,\vec{x})^T \to T_0\,(t,\vec{x})^T = (-t,\vec{x})^T \ . \tag{2.22}$$

Secondly, we have the parity operator $\mathscr{P}$ that changes the sign of the spacial coördinates

$$\mathscr{P} : (t,\vec{x})^T \to P_0\,(t,\vec{x})^T = (t,-\vec{x})^T \ . \tag{2.23}$$

The transformation matrices for these transformations are given by

$$T_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.24}$$

These transformations can be combined to give the $\mathscr{P}\mathscr{T}$-transformation that is given by

$$\mathscr{P}\mathscr{T} : \mathbf{x} = (t,x,y,z)^T \to P_0 T_0\,(t,x,y,z) = (-t,-x,-y,-z)^T = -\mathbf{x} \,. \tag{2.25}$$

In the previous section we discussed the proper Lorenz transformations. Note that in equation (2.9) we assumed that the parameters $w^{\alpha\beta}$ are real. If we allow the space-time coördinates to be complex, one can extend the parameters $w^{\alpha\beta}$ to the complex plane

$$w^{\alpha\beta} \in \mathbb{C} \ . \tag{2.26}$$

In this way, the $\mathscr{PT}$-transformation can be obtained from the Lorenz transformations along the complex plane. One can see this by first considering a boost in the x-direction with angle $i\pi$

$$\Lambda_x(i\pi) : \mathbf{x} \to \Lambda_x(i\pi)\mathbf{x} = (-t, -x, y, z)^T \ , \tag{2.27}$$

that switches the sign of the t- and x-coördinate. A similar boost along the y-direction will change the sign of the t- and y-coördinate

$$\Lambda_y(i\pi) : \mathbf{x} \to \Lambda_y(i\pi)\mathbf{x} = (-t, x, -y, z)^T \ , \tag{2.28}$$

and for such a boost along the z direction, one finds

$$\Lambda_z(i\pi) : \mathbf{x} \to \Lambda_z(i\pi)\mathbf{x} = (-t, x, y, -z)^T \ . \tag{2.29}$$

The $\mathscr{PT}$-transformation can thus be obtaind by applying these three transformations after each other

$$\mathscr{PT} : \mathbf{x} = (t, x, y, z)^T \to \Lambda_z(i\pi)\Lambda_y(i\pi)\Lambda_x(i\pi)\mathbf{x} = (-t, -x, -y, -z)^T \ . \tag{2.30}$$

A more in dept discussion on this can be found in [63]. Both continuous and discrete symmetries play an important role in the study of Quantum systems. The existence of continuous symmetries implies the existence of a conserved current know as the Noether current. The proof of this does however rely on the Lagrangian being Hermitian. In this will work we will be interested in studying systems that have a discrete $\mathscr{PT}$-symmetry instead of having a Hermitian symmetry.

2.2 Non-Hermitian Systems

One of the first conditions that are imposed when studying Quantum systems is Hermicity. This limits us, from the start, into only considering those operators that are Hermitian. This restriction ensures that the set of observable energies will be real and bounded from bellow and that the Quantum system possesses unitary time evolution. This latter condition of unitary time evolution is there so that the expected result of a measurement of a state will not change over time.

Remark that all of these conditions, except for Hermicity, are physically motivated restrictions that one can put on a system. Hermicity, however, is a mathematical condition and does not have a clear physical meaning. It is well known that Hermicity does provide these other conditions mentioned here, but one can wonder if it is necessary to have them. In other words, it raises the question of whether a theory truly needs to be Hermitian to be physical. If the answer to this question were to be no, the logical next step would be to look at other conditions, preferably physically motivated ones, that would also ensure these conditions.

This question has been studied extensively in works such as [1–7, 43, 46, 64–66] where it has been shown that Hermicity can indeed be replaced by the condition of having unbroken $\mathscr{P}\mathscr{T}$-symmetry. In the following sections, we will delve deeper into what this condition exactly entails.

2.2.1 Unbroken $\mathscr{P}\mathscr{T}$-symmetry

A system is said to posses unbroken $\mathscr{P}\mathscr{T}$-symmetry if the system is both $\mathscr{P}\mathscr{T}$-symmetric and the eigenstates of the Hamiltonian $\hat{H}$ are also eigenstates of the $\mathscr{P}\mathscr{T}$-opperator. In this section we will prove that this condition is indeed sufficient to have real energies.

The discrete transformations $\mathscr{P}$, $\mathscr{T}$ and $\mathscr{P}\mathscr{T}$ are all involutory transformations

$$\mathscr{P}^2 = \mathscr{T}^2 = (\mathscr{P}\mathscr{T})^2 = id \,, \tag{2.31}$$

and commute

$$[\mathscr{P}, \mathscr{T}] = 0 \,. \tag{2.32}$$

From this, one can easily see that the eigenvalues of the $\mathscr{P}\mathscr{T}$-transformation must be on the complex unit circle,

$$\mathscr{P}\mathscr{T}\left(\phi\right) = \lambda\phi \ \text{, with } |\lambda|^2 = 1 \ . \tag{2.33}$$

With this in mind, the proof that the energies of an unbroken $\mathscr{P}\mathscr{T}$-symmetric system has real energies is straightforward.

Proof. Let ϕ be an eigenstate of the Hamiltonian $\hat{H}$, with energy E,

$$\hat{H}\phi = E\phi \ . \tag{2.34}$$

Since $\hat{H}$ has an unbroken $\mathscr{P}\mathscr{T}$-symmetry, ϕ is also an eigenvector of the $\mathscr{P}\mathscr{T}$-operator, with eigenvalue λ. From equation (2.34) it then follows that

$$\mathscr{P}\mathscr{T}\left(\hat{H}\phi\right) = \left(\mathscr{P}\mathscr{T}\right)E\left(\mathscr{P}\mathscr{T}\right)^2\phi = E^*\lambda\phi = \lambda E^*\phi \ , \tag{2.35}$$

where we used that $\mathscr{P}\mathscr{T}$ is an anti-linear transformation. Since the $\mathscr{P}\mathscr{T}$-operator is a symmetry of $\hat{H}$, it follows that $[\mathscr{P}\mathscr{T},\hat{H}] = 0$. Equation (2.35) should thus also be equal to

$$\hat{H}\left(\mathscr{P}\mathscr{T}\right)\phi = \hat{H}\lambda\phi = \lambda\hat{H}\phi = \lambda E\phi \ . \tag{2.36}$$

Using this, it is straightforward to show that

$$\left[\mathscr{P}\mathscr{T},\hat{H}\right]\phi = \lambda\left(E^* - E\right)\phi = 0 \ , \tag{2.37}$$

and thus

$$E = E^* \tag{2.38}$$

□

This proof shows that any theory with an unbroken $\mathscr{P}\mathscr{T}$-symmetry also has a real spectrum. However, to have a consistent physical theory, one also needs to have a unitary time evolution.

Unitary time evolution ensures that probabilities of measurments stay constant in time. For Hermitian systems, the state $|\phi\rangle$ and its conjugate $\langle\phi|$ evolves over a time t as

$$\langle\phi| \to \langle\phi|e^{-it\hat{H}} \quad \text{and} \quad |\,\phi\rangle \to e^{it\hat{H}}|\phi\rangle \ . \tag{2.39}$$

The probability of measuring such a state is then given by

$$\langle\phi|e^{-it\hat{H}}e^{it\hat{H}}|\phi\rangle = \langle\phi|\phi\rangle \ , \tag{2.40}$$

and thus independent of the time t. Remark that the relation between the evolution of the state and conjugate state as defined in equations (2.39) depends on the definition of the inner product of our Hilbert space. As a consequence, the definition of unitarity should also depend on this inner product. A non-Hermitian Hamiltonian will thus in general not be unitary under a Hermitian inner product. In order to have a unitary time evolution, one needs to be careful when defining a consistent inner product.

2.2.2 Left and Right Eigenfunction of non-Hermitian Systems

We have shown that unbroken $\mathscr{P}\mathscr{T}$-symmetric systems have real energies despite not being Hermitian. Non-Hermitian systems are different from Hermitian ones in how the left and right eigenfunctions are related. Because of this, one needs to take care in defining the inner product to define an orthonormal basis. We can show this with an example by considering a non-Hermitian matrix of the form

$$A = \begin{pmatrix} a & c \\ -c & b \end{pmatrix} \quad \text{, with } \ a,b,c \in \mathbb{R} \ . \tag{2.41}$$

This matrix is symmetrical under the discrete transformation

$$A = \begin{pmatrix} a & c \\ -c & b \end{pmatrix} \to \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]^T = A \ . \tag{2.42}$$

Thus the $\mathscr{PT}$-transformation that leaves A invariant acts on the vectors as

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} . \tag{2.43}$$

The eigenvalues of A are given by,

$$\lambda_A^{\pm} = \frac{a+b}{2} \pm \frac{\sqrt{(a-b+2c)(a-b-2c)}}{2} . \tag{2.44}$$

These eigevalues are real as long as

$$\frac{|a-b|}{2} \geq c . \tag{2.45}$$

Within this region, the system is in the unbroken $\mathscr{PT}$-symmetric phase and the eigenvalues are all real. Unlike a Hermitian system, this matrix has left and right eigenvectors that are not related by complex conjugation. Instead, the eigenvectors are given by

$$\mathbf{e}_{A,L}^{\pm} = N_L^{\pm} \begin{pmatrix} (a-b) \pm \sqrt{(a-b)^2 - 4c^2} \\ 2c \end{pmatrix}^T , \ \mathbf{e}_{A,R}^{\pm} = N_R^{\pm} \begin{pmatrix} 2c \\ (b-a) \pm \sqrt{(a-b)^2 - 4c^2} \end{pmatrix} , \tag{2.46}$$

where $\mathbf{e}_{A,L}^{\pm}$ are the left eigenvectors and $\mathbf{e}_{A,R}^{\pm}$ are the right eigenvectors. The factors $N_L^{\pm}$ and $N_R^{\pm}$ are suitable normalization constants that we will fix later. Remark how the non-Hermitian coupling c changes the behaviour as compared to a Hermitian system:

- $\left(\mathbf{e}_{A,L}^{\pm}\right)^{\dagger} \neq \mathbf{e}_{A,R}^{\pm}$
- $\left(\mathbf{e}_{A,R}^{\pm}\right)^{\dagger} \cdot \mathbf{e}_{A,R}^{\mp} = \mathbf{e}_{A,L}^{\pm} \cdot \left(\mathbf{e}_{A,L}^{\mp}\right)^{\dagger} = 8c^2 \neq 0.$

However one does observe for the vectors in (2.46) the relations:

- $\left[\mathbf{e}_{A,R}^{\pm}\right]^{\mathscr{PT}} = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{e}_{A,R}^{\pm}\right]^T \propto \mathbf{e}_{A,L}^{\pm}$
- $\mathbf{e}_{A,L}^{\pm} \cdot \mathbf{e}_{A,R}^{\mp} = 0.$

A natural choice for a suitable inner product, that would lead to a consistent theory, could thus be the $\mathscr{P}\mathscr{T}$-inner product of the form

$$\langle \mathbf{s}^{\mathscr{P}\mathscr{T}} | \mathbf{l} \rangle = (\mathscr{P}\mathscr{T}(\mathbf{s}))^T \mathbf{l} = \mathbf{s}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{l} \,. \tag{2.47}$$

Under such an inner product, we can fix the normalization constants $N_L^{\pm}$ and $N_R^{\pm}$ as

$$N_L^{\pm} = \theta(a-b) \cdot N_R^{\mp} = \left(2\sqrt{(a-b)^2 - 4c^2} \left| (a-b) \pm \sqrt{(a-b)^2 - 4c^2} \right| \right)^{-1/2} \,, \tag{2.48}$$

where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} -1 & x < 0 \\ +1 & x > 0 \,. \end{cases} \tag{2.49}$$

This ensures that

$$\mathscr{P}\mathscr{T} \left(\mathbf{e}_{A,L}^{\pm} \right)^T = \mathbf{e}_{A,R}^{\pm} \quad \text{and} \quad \mathbf{e}_{A,L}^{\pm} \cdot \mathbf{e}_{A,R}^{\pm} = \pm \theta\left(a-b\right) \,. \tag{2.50}$$

Even though this inner product seems a very straightforward and elegant generalisation of the inner product used in Hermitian theories, it is still not sufficient to have a physical theory. Under the $\mathscr{P}\mathscr{T}$-inner product, eigenvectors belonging to different eigenvalues are orthogonal, but the norms of these eigenvectors are not always positive.

This inner product is however useful in that it highlights another, previously hidden symmetry of these systems. Under the $\mathscr{P}\mathscr{T}$-inner product, half the eigenstates do have a positive norm while the other half possesses a negative norm. One can exploit this apparant symmetry by defining a linear opperator $\mathscr{C}'$ that acts on the eigenstates as

$$\mathscr{C}' \left(\mathbf{e}_{A,L}^{\pm} \right) = \begin{cases} +\mathbf{e}_{A,L}^{\pm} & \text{if} \quad |\mathbf{e}_A^{\pm}|^2_{\mathscr{P}\mathscr{T}} > 0 \\ -\mathbf{e}_{A,L}^{\pm} & \text{if} \quad |\mathbf{e}_A^{\pm}|^2_{\mathscr{P}\mathscr{T}} < 0 \,. \end{cases} \tag{2.51}$$

This allows us to define a physcially consistent inner product of the form

$$\langle \mathbf{s}^{\mathscr{C}'\mathscr{P}\mathscr{T}} | \mathbf{l} \rangle = \mathscr{C}'\mathscr{P}\mathscr{T}(\mathbf{s})^T \cdot \mathbf{l}\,, \tag{2.52}$$

for which the eigenvectors of $\hat{H}$ form an orthonormal basis. The normalizations as described in (2.48) are still sufficient for this inner product. Remark also that under such a inner product one still is asured a unitary time evolution since $\hat{H}$ is also $\mathscr{C}'\mathscr{P}\mathscr{T}$ symmetric

$$\langle \mathbf{l}|\mathbf{l}\rangle_{\mathscr{C}'\mathscr{P}\mathscr{T}} \to \langle \mathbf{l}|e^{-it\hat{H}}e^{it\hat{H}}|\mathbf{l}\rangle_{\mathscr{C}'\mathscr{P}\mathscr{T}} = \langle \mathbf{l}|\mathbf{l}\rangle_{\mathscr{C}'\mathscr{P}\mathscr{T}}\,. \tag{2.53}$$

It turns out that the unbroken $\mathscr{P}\mathscr{T}$-symmetric systems can indeed describe a physical theory. The unbroken $\mathscr{P}\mathscr{T}$-symmetric nature of our theory ensures us that all the energies of our system are real. Unitary time evolution is also ensured given that we equip out theory with a $\mathscr{C}'\mathscr{P}\mathscr{T}$-inner product. The construction of this inner product highlights another hidden symmetry in these unbroken $\mathscr{P}\mathscr{T}$-symmetric systems. This symmetry is called $\mathscr{C}'$. Remark, however, that this transformation is different from the charge conjugation transformation that we are familiar with in Quantum Field Theory. This $\mathscr{C}'$-opperator is discussed in [1, 65].

2.3 Non-Hermitian Quantum Mechanical Systems

2.3.1 Non-Hermitian Model and Spectrum

In the previous section, we discussed how the condition of Hermicity can be replaced by unbroken $\mathscr{P}\mathscr{T}$-symmetry. Non-Hermitian Quantum Mechanical system have been extensively been studied in works such as [1, 2, 24–28, 32, 67]. In this section, we will show how this happens for a specific Quantum Mechanical system. Later, in Chapter 3, we will discuss in detail how this translates to a Quantum Field Theory. For now, we focus our attention on a family of systems described by the Hamiltonians $\hat{H} = \hat{p}^2 + \hat{x}^2\,(i\hat{x})^{\varepsilon}$, with $\varepsilon \in \mathbb{R}$. This particular system has been discussed in works such as [1, 2, 4–6, 24].

Consider the set of Hamiltonians of the form

$$\hat{H} = \hat{p}^2 + \hat{x}^2 \left(i\hat{x}\right)^{\varepsilon} \quad \text{with} \quad \varepsilon \in \mathbb{R} \ . \tag{2.54}$$

For general values of ε, these Lagrangians are not Hermitian. They are however still invariant under a $\mathscr{P}\mathscr{T}$ transformation of the form

$$\mathscr{P}: \begin{cases} p \to -p \\ x \to -x \end{cases} \quad \text{and} \quad \mathscr{T}: \begin{cases} p \to -p \\ i \to -i \end{cases} \ . \tag{2.55}$$

The problem of finding the energies of these systems can be reduced into solving the eigenvalue problem

$$\hat{H}\phi\left(x\right) = -\phi''(x) + x^2 \left(ix\right)^{\varepsilon} \phi\left(x\right) = E\phi\left(x\right) \ . \tag{2.56}$$

The energies of this equation can be found nummerically. These are given in figure 2.1. This plot highlights three different regions as a function of ε. Firstly for the region $\varepsilon < -1$, there are no real energies for which a solution of (2.56) exists. All the energies are complex and appear in complex conjugate pairs. Secondly, in the region $-1 < \varepsilon < 0$, there are a finite amount of real energies and an infinite amount of complex conjugate solutions. These complex energies appear again in complex conjugate pairs. The amount of real energy solutions increases when ε increases. When $-1 \leq \varepsilon \leq -0.57793$, the only real energy level is the groundstate energy. As $\varepsilon \to -1^+$ the groundstate energy diverges to $+\infty$.

The last region, $\varepsilon \geq 0$, is where all the energies of our system are real. This is the region of unbroken $\mathscr{P}\mathscr{T}$-symmetry. We see that the energy levels rise with increasing ε. The lower bound of the region, $\varepsilon = 0$, corresponds to the Hermitian harmonic oscillator, whose energy levels are given by $E_n = (2n+1)$.

2.3.2 Asymptotic Behaviour

The problem of solving the eigenvalue equation (2.56) is more complex then one would expect from the Hermitian case. Remember that when solving an eigenvalue

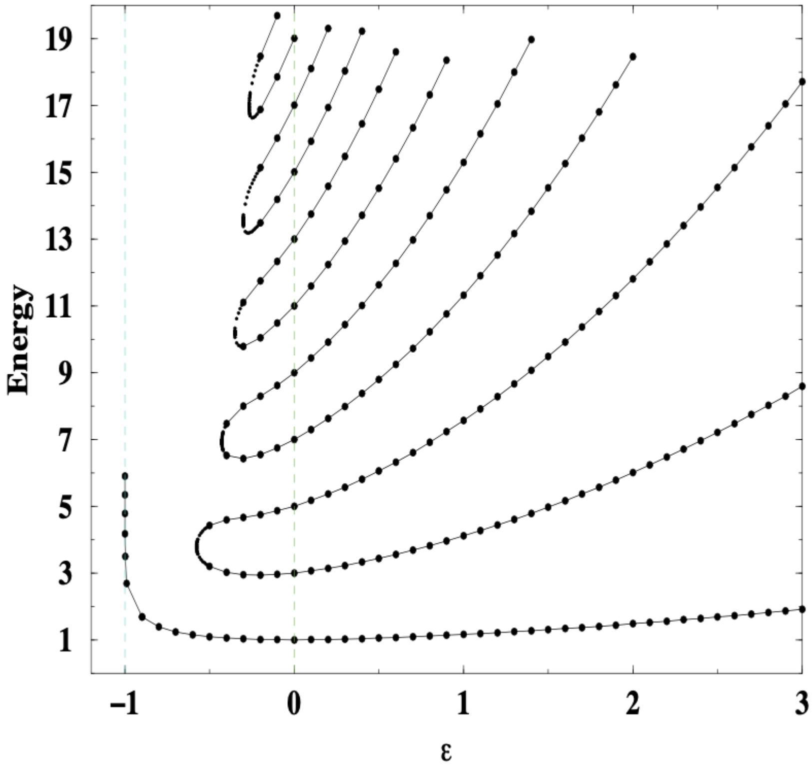

Figure 2.1: The energy levels of the Hamiltonian $H = \hat{p}^2 + \hat{x}^2 (i\hat{x})^\varepsilon$ with respect to ε. The plot shows three regions in parameter space. Firstly, when $\varepsilon < -1$, no values are plotted since no real energies exist. Secondly, when $-1 < \varepsilon < 0$ the system has a finite amount of real energies. When $\varepsilon \to -1^+$ the groundstate energies diverge. In this region, the groundstate energy decreases with increasing ε. When $0 < \varepsilon$ the spectrum is entirely real and positive and the ground state energies rise with increasing ε. This figure has been taken from the work from Carl M. Bender, *Making Sense of Non-Hermitian Hamiltonians*, Reports on Progress in Physics 70 nr.6, (2007): 947-1018. [1]. Reprint with permission from author. ©IOP Publishing. Reproduced with permission. All rights reserved

problem of the form

$$\hat{H}\phi = E\phi \ , \tag{2.57}$$

in the Hermitian case, one needs to fix the boundary such that $\phi(x) \to 0$ as $|x| \to \infty$ along the real axis. For an eigenvalue problem of the form (2.56) the same boundary conditions can only hold as long as $-1 < \varepsilon < 2$. For arbitrary values of ε this is however not the case. In order to describe meaningful boundrary conditions, we will need to extend the eigenvalue problem (2.56) into the complex plane. The

WKB approximation can then be used to fix the boundary conditions and later to estimate the energies. The eigenvalue problem (2.56) can be writen as

$$\phi''(x) = \left[x^2 (ix)^\varepsilon - E\right] \phi(x) \, . \tag{2.58}$$

Using the WKB approximation, the solution for $\phi(x)$ in the limit $|x| \to \infty$ is given by

$$\phi(x) \approx \exp\left(-\int_x ds\sqrt{s^2 (is)^\varepsilon - E}\right) \approx \exp\left(-\int_x ds\sqrt{s^2 (is)^\varepsilon}\right) \, . \tag{2.59}$$

Solving this integral in this limit is straightforward and gives

$$\phi(x) \approx \exp\left(-\frac{r^{2+\varepsilon/2}}{2+\varepsilon/2} e^{i\theta[2+\varepsilon/2]+i\pi\varepsilon/4}\right) \quad \text{when} \;\; r \to \infty \, , \tag{2.60}$$

where $x = re^{i\theta}$. One can thus see that $\phi(x)$ only decreases exponentially fast to 0 in the limit $r \to +\infty$ if

$$\theta \to -\frac{\varepsilon}{4+\varepsilon}\frac{\pi}{2} \tag{2.61a}$$

$$\theta \to +\frac{\varepsilon}{4+\varepsilon}\frac{\pi}{2} - \pi \, . \tag{2.61b}$$

Such a convergens to 0 still holds, even though not exponentially fast, when

$$-\frac{\pi}{2} < \theta\left(\frac{4+\varepsilon}{2}\right) + \frac{\pi}{4}\varepsilon < \frac{\pi}{2} \tag{2.62a}$$

$$-\frac{3\pi}{2} < \theta\left(\frac{4+\varepsilon}{2}\right) + \frac{\pi}{4}\varepsilon < -\frac{\pi}{2} \, , \tag{2.62b}$$

thus the opening angles wherein convergence still holds are

$$\Delta\theta = \frac{2\pi}{4+\varepsilon} \, . \tag{2.63}$$

These regions of convergence are plotted in figure 2.2. On this graph the wedges wherein the convergence $\phi(x) \to 0$ for $|x| \to \infty$ holds are shown by the grey region.

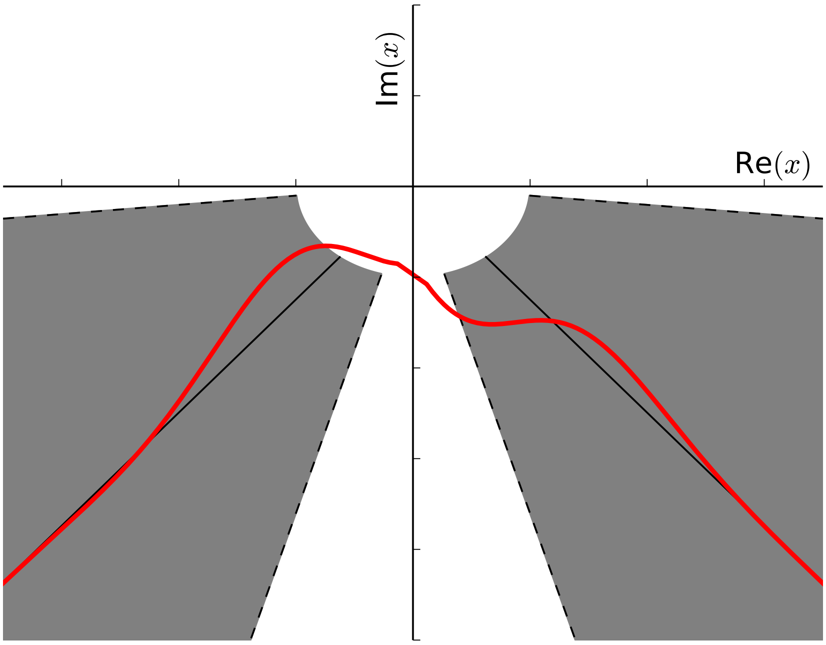

Figure 2.2: The plot shows the wedges in the complex-x plane wherein $\phi(x) \to 0$ when $|x| \to +\infty$ for a system with $\varepsilon = 2.2$. Along the black straight line this convergence happens exponentially fast. The red line is a path along which we can integrate the field $\phi(x)$.

The boundary of these regions is shown by the black dashed lines. Within this these wedges, convergence along the full black line will be exponentially fast.

2.3.3 Classical Path of the System

The classical equations of motion for a Hamiltonian of the form of equation (2.54) are given by

$$\begin{cases} \frac{dx}{dt} & = \frac{\partial H}{\partial p} = 2p \\ \frac{dp}{dt} & = -\frac{\partial H}{\partial x} = i\left(2+\varepsilon\right)\left(ix\right)^{1+\varepsilon} \end{cases} , \tag{2.64}$$

from which we find that

$$\frac{d^2x}{dt^2} = 2i\left(2+\varepsilon\right)\left(ix\right)^{1+\varepsilon} . \tag{2.65}$$

This equation can be solved and we obtain that

$$\frac{1}{2}\frac{dx}{dt} = \pm\sqrt{E + (ix)^{2+\varepsilon}}\,. \tag{2.66}$$

Remark that this is a complex continuation of the classical equations. Even though $x(t)$ is complex, one can still treat t as a real variable. Equation (2.66) shows that the turning points of our system are given by the roots of the function

$$f(x) = \sqrt{E + (ix)^{2+\varepsilon}}\,. \tag{2.67}$$

We use the notation $x_\pm$ for those roots of (2.67) that continue off the real axis as ε diverges from 0,

$$x_- = E^{\frac{1}{2+\varepsilon}} e^{i\pi\left(\frac{4+3\varepsilon}{4+2\varepsilon}\right)} \quad \text{and} \quad x_+ = E^{\frac{1}{2+\varepsilon}} e^{-i\pi\left(\frac{\varepsilon}{4+2\varepsilon}\right)}\,. \tag{2.68}$$

The energy level can be estimated using the WKB approximation. The leading order WKB phase integral quantization condition is given by

$$(n+1/2)\,\pi = \int_{x_-}^{x_+} dx \sqrt{E - x^2\,(ix)^{\varepsilon}} \tag{2.69}$$

For positive ε, we can deform the phase-integral contour so that it follows the rays from x_- to 0 and from 0 to x_+. With this in mind, we then find

$$(n+1/2)\,\pi = 2\sin\left(\frac{\pi}{\varepsilon+2}\right) E^{\left(\frac{4+\varepsilon}{4+2\varepsilon}\right)} \int_0^1 ds \sqrt{1 - s^{\varepsilon+2}}\,, \tag{2.70}$$

and thus

$$E_n \approx \left[\frac{\Gamma\left(\frac{8+3\varepsilon}{4+2\varepsilon}\right)\sqrt{\pi}\,(n+1/2)}{\sin\left(\frac{\pi}{2+\varepsilon}\right)\Gamma\left(\frac{3+\varepsilon}{2+\varepsilon}\right)}\right]^{\frac{4+2\varepsilon}{4+\varepsilon}}\,. \tag{2.71}$$

We are now able to describe the classical paths for systems of the form (2.54) in the case where $\varepsilon \in \mathbb{N}$. A detailed description that also includes the case where ε is a non-integer can be found in works such as [1, 5, 24].

The first case we discuss is where $\varepsilon = 0$. As discussed before, the system is then Hermitian and describes a Harmonic oscillator [68]. For such a system the classical path is an oscillation between the two turning points,

$$t_1 = \sqrt{E} \quad , \quad t_2 = -\sqrt{E} \; . \tag{2.72}$$

Remark that the solutions we are looking at, are these solutions to equation (2.65) for a particular fixed value of E. To describe a solution to equation (2.65), we need to specify an initial condition. We know from the Hermitian case, that when we have an initial position x_0 on the x-axis, between the turning points,

$$-\sqrt{E} \leq x_0 \leq \sqrt{E} \; , \tag{2.73}$$

the path goes towards a turning point, returns towards the other turning point, and oscillates between these two points. When we describe this Hermitian system and only allow x to be real, no solution exists for initial values outside these turning points. This can be seen from equation (2.66) since such an initial condition would imply an imaginary initial velocity. Since we eventually want to deal with non-Hermitian theories, it does make sense to also allow for a complex continuation of the paths. When this is the case, paths with initial conditions outside the turning points can exist. Some of these solutions have been plotted in figure (2.3).

In this plot, the turning points are given by the red dots. The red line connecting these dots is the Hermitian path that lies on the real axis. Since we also allow for the paths to be complex, it is also possible to find solutions that cross the real x-axis outside the turning points. Such paths form ellipses with foci on the turning points. Remark that none of these paths can physically cross. Remark also that the period of these ellipses and the classical path are all the same. This is a straightforward consequence of the Cauchy theorem. Finally, we remark that all these paths are $\mathscr{P}\mathscr{T}$-symmetric solutions.

We can now look at the solutions when $\varepsilon = 1$. In this case the system has three

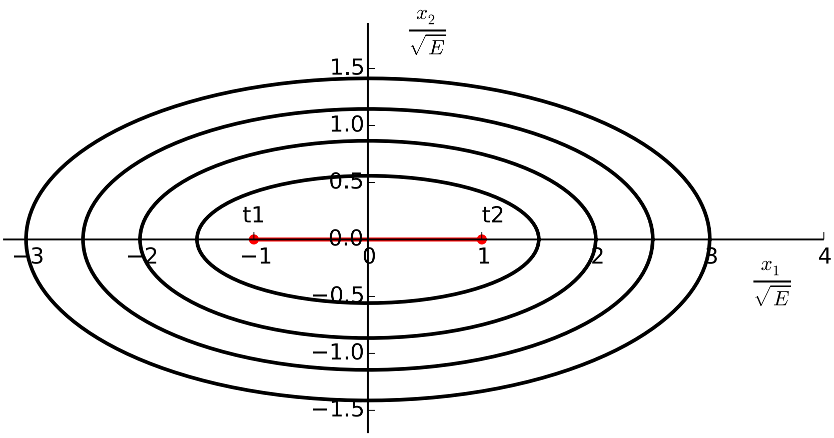

Figure 2.3: The plot shows different classical paths that are solutions to the Hamiltonian $\hat{H} = \hat{p}^2 + \hat{x}^2$, where we allow for complex continuations of the paths $\hat{x} = x_1 + ix_2$. The turning points for this system t_1, t_2 are shown by the red dots in the plot. The red line connecting these two lines is the normal Hermitian path. The other paths form ellipses around these two turning points. Remark that none of the paths intersect.

turning points given by

$$t_1 = i\sqrt[3]{E} \quad , \quad t_2 = e^{i\frac{7}{6}\pi}\sqrt[3]{E} \quad , \quad t_3 = e^{i\frac{11}{6}\pi}\sqrt[3]{E} \; . \tag{2.74}$$

One can see that these points form a $\mathscr{P}\mathscr{T}$-symmetric configuration on the complex plane. We can again first look at the paths that start at these turning points. The first path connects the turning points $t2$ and $t3$ through an arc below the real axis. After it reaches the point $t3$, it returns to $t2$ and oscillates between those two turning points, similar what we saw in the case where $\varepsilon = 0$. The path that starts at the turning point t_1 goes up towards the positive imaginary axis $+i\infty$. This happens in a finite amount of time $\sqrt{\pi}\,\Gamma\left(\frac{4}{3}\right)/\Gamma\left(\frac{5}{6}\right)$. Other paths follow periodic orbits around the turning points t_2 and t_3. None of the paths intersect. This means that these orbits are pinched under the point $x1$. These indentations become sharper for larger orbits. All these orbits around the t_2 and t_3 turning points, and the orbit connecting these points, have the same period $2\sqrt{3\pi}\,\Gamma\left(\frac{4}{3}\right)/\Gamma\left(\frac{5}{6}\right)$. These paths have been plotted in figure (2.4).

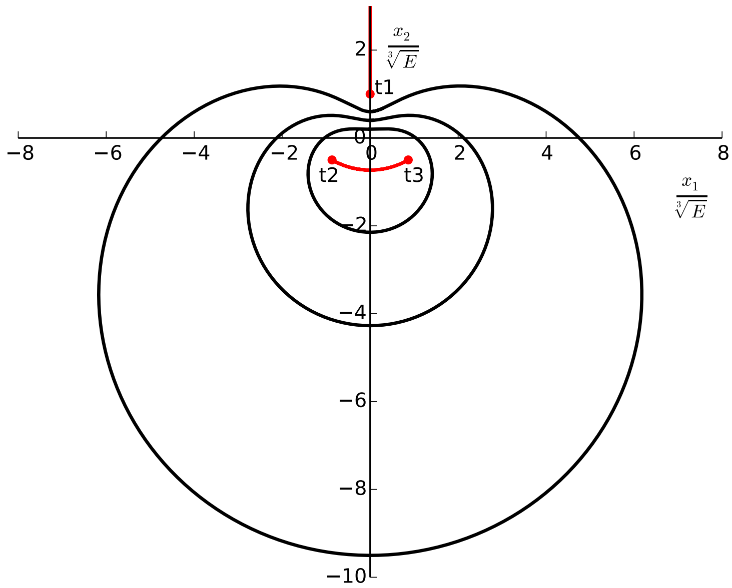

Figure 2.4: The plot shows different classical paths that are solutions to the Hamiltonian $\hat{H} = \hat{p}^2 + i\hat{x}^3$, where we allow for complex continuations of the paths $\hat{x} = x_1 + ix_2$. The turning points for this system t_1, t_2, t_3 are shown by the red dots in the plot. The red lines are the paths that originate from one of the turning points. The other paths orbit around these two turning points t_2 and t_3. Remark that none of the paths intersect.

The last case we explicitly plot is where $\varepsilon = 2$. In this case the system has four turning points

$$t_1 = \sqrt[4]{E}e^{i\frac{3\pi}{4}} \quad , \quad t_2 = \sqrt[4]{E}e^{i\frac{\pi}{4}} \quad , \quad t_3 = \sqrt[4]{E}e^{i\frac{5\pi}{4}} \quad , \quad t_4 = \sqrt[4]{E}e^{i\frac{7\pi}{4}} \; . \tag{2.75}$$

Different paths for this system have plotted in figure (2.5). Similar to the previous plots, the paths starting at one turning point will go towards another turning point and return. After this, it will again oscillate between these two points. Other paths

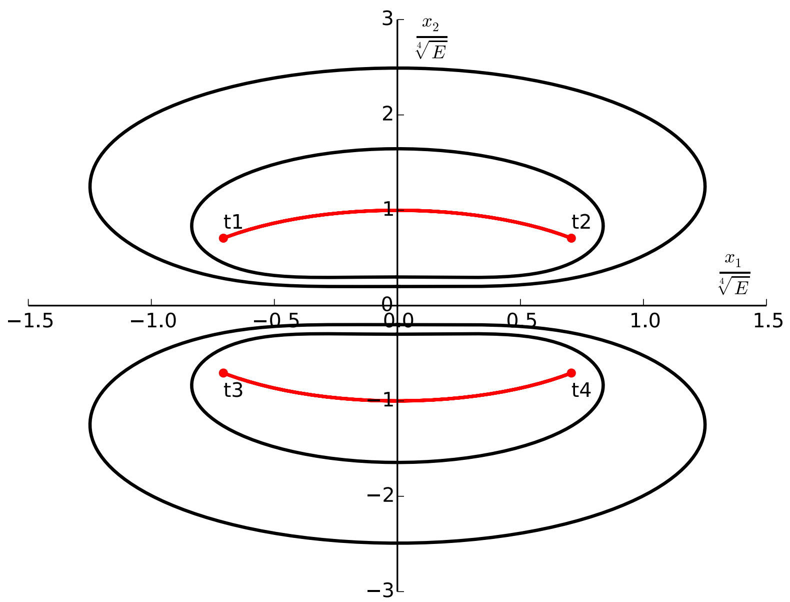

Figure 2.5: The plot shows different classical paths that are solutions to the Hamiltonian $\hat{H} = \hat{p}^2 - \hat{x}^4$, where we allow for complex continuations of the paths $\hat{x} = x_1 + ix_2$. The turning points for this system t_1, t_2, t_3 and t_4 are shown by the red dots in the plot. The red lines are the paths that originate from the turning points. The other paths orbit around the two turning points t_1 and t_2 or the turning points t_3 and t_4. Remark that none of the paths intersect.

will orbit around the these turning points, all with the same time period. In this case, where $\varepsilon = 2$, the time is given by $2\sqrt{2\pi}\,\Gamma\left(\frac{5}{4}\right)/\Gamma\left(\frac{3}{4}\right)$. Once more, remark that the paths again do not intersect and they are all $\mathscr{PT}$-symmetric.

We can generalise these examples to any values of $\varepsilon \in \mathbb{N}$. Remark that the configuration of turning points, as well as all the paths, all appear in a $\mathscr{PT}$-symmetric manner. For ε an integer, the turning points are given by

$$t_k = \sqrt[\varepsilon+2]{E}e^{i\pi\left(\frac{4k-\varepsilon+4}{2(2+\varepsilon)}\right)} \quad , \text{ with } k \in \{1, 2, \ldots 2+\varepsilon\} \; . \tag{2.76}$$

If ε is even, every turning point is the image of the $\mathscr{PT}$-transformation of another turning point. Any path starting at a turning point will go towards the $\mathscr{PT}$-image of that turning point and return. Afterwards, it will oscillate between these two

points. Other paths will follow circular paths around two such points.
If ε is an odd turning point, then one turning point will be its own image under a $\mathscr{PT}$-transformation. Such a turning point will always be on the imaginary axis. The path starting from this turning point will go along the imaginary axis towards $\pm i\infty$. Other paths will be similar to those discussed when ε is even. Remark that none of the paths can cross. This means that a path that does not start at an imaginary turning point can cross the real axis at any point, but it can only cross the imaginary axis where no other path exists.

2.4 Conclusion

In this chapter, we discussed both continuous and discrete transformations and symmetries and their role in physical systems. We have discussed the Noether's theorem and have shown that its derivation uses Hermicity. We then introduced non-Hermitian, but unbroken $\mathscr{PT}$-symmetric models. We have shown that these systems can still be physically relevant despite not being Hermitian. We were able to show that the energies of these systems where real and that, given a suitable inner product, they could also guarantee unitary time evolution.
We focused our attention to a particular set of such Quantum Mechanical systems of the form

$$\hat{H} = \hat{p}^2 + \hat{x}^2 (i\hat{x})^{\varepsilon} \quad , \ \text{with } \varepsilon \in \mathbb{R} \, . \tag{2.77}$$

For these systems, we discussed the energies and the conditions for which they are real. We discussed the conditions on how to numerically find the energies, in particular the boundary condition for such systems. Lastly, we discussed the classical path of these systems for $\varepsilon \in \mathbb{N}$. For an even ε, the paths revolve around two $\mathscr{PT}$-conjugate turning points. When ε is odd, another turning point that is $\mathscr{PT}$ invariant and goes to $\pm i\infty$ excists.

Chapter 3

Symmetries and Conserved Currents in non-Hermitian Quantum Field Theories

3.1 Non-Hermitian Scalar Field Theory

The first non-Hermitian model we study is given by the Lagrangian of the form

$$\mathscr{L}_s = \begin{pmatrix} \partial_\alpha \phi_1^* & \partial_\alpha \phi_2^* \end{pmatrix} \begin{pmatrix} \partial^\alpha \phi_1 \\ \partial^\alpha \phi_2 \end{pmatrix} - \begin{pmatrix} \phi_1^* & \phi_2^* \end{pmatrix} M^2 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} , \tag{3.1}$$

where $M^2 \neq \left(M^2\right)^\dagger$. The non-Hermitian mass term squared M^2 is taken to have the form

$$M^2 = \begin{pmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2 \end{pmatrix} , \tag{3.2}$$

where we assume that $m_1^2, m_2^2, \mu^2 \in \mathbb{R}$ and $m_1^2, m_2^2 > 0$. The anti-Hermitian terms of these Lagrangians are proportional to μ^2 and the Hermitian limit of these systems can be obtained by taking the limit $\mu^2 \to 0$. This system has extensively been studied in works such as [69, 70] and most of this chapter is based on the work covered in those papers.

3.1.1 Discrete Symmetries

We start the study of this non-Hermitian Quantum Field Theory by discussing the discrete transformations of this system. The parity transformation acts in general on a scalar field as

$$\mathscr{P} : \phi(t,\vec{x}) \to \phi'(t,-\vec{x}) = e^{i\gamma_1}\phi(x) \quad ,\text{with} \quad \gamma_1 \in \mathbb{R} \, , \tag{3.3}$$

and is thus uniquely defined up to a phase factor. The $\mathscr{P}$-opperator acts thus on the scalar fields of our system as

$$\mathscr{P} : \begin{pmatrix} \phi_1(t,\vec{x}) \\ \phi_2(t,\vec{x}) \end{pmatrix} \to \begin{pmatrix} \phi_1'(t,-\vec{x}) \\ \phi_2'(t,-\vec{x}) \end{pmatrix} = \begin{pmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{i\alpha_2} \end{pmatrix} \begin{pmatrix} \phi_1(t,\vec{x}) \\ -\phi_2(t,\vec{x}) \end{pmatrix} \, , \tag{3.4}$$

where $\alpha_i \in \mathbb{R}$. The $\mathscr{T}$-transformation in turn acts on a scalar field as

$$\mathscr{T} : \phi(t,\vec{x}) \to \phi'(-t,\vec{x}) = e^{i\gamma_2}\phi^*(x) \quad ,\text{with} \quad \gamma_2 \in \mathbb{R} \, , \tag{3.5}$$

again defined up to a phase factor. The $\mathscr{T}$-transformation acts on our system as

$$\mathscr{T} : \begin{pmatrix} \phi_1(t,\vec{x}) \\ \phi_2(t,\vec{x}) \end{pmatrix} \to \begin{pmatrix} \phi_1'(-t,\vec{x}) \\ \phi_2'(-t,\vec{x}) \end{pmatrix} = \begin{pmatrix} e^{i\beta_1} & 0 \\ 0 & e^{i\beta_2} \end{pmatrix} \begin{pmatrix} \phi_1^*(t,\vec{x}) \\ \phi_2^*(t,\vec{x}) \end{pmatrix} \, , \tag{3.6}$$

with $\beta_i \in \mathbb{R}$.

Systems with a non-Hermitian potential seem to interact with an environment. This interaction manifests itself into the presence of source and sinks in the system. When we have a $\mathscr{P}\mathscr{T}$-symmetric system, there is as much gained from the sources as there is lost from the sinks. Under a $\mathscr{T}$-operator, the flow of time changes direction. Thus, the sinks in our systems now become sources, and the sources become sinks. Exchanging the sources and sinks in our system would correspond to changing the sign of μ^2. From this it follows that $\beta_1 = \beta_2$. Since the Lagrangian (3.1) is $\mathscr{P}\mathscr{T}$-symmetric, it also follows that $\alpha_1 = \alpha_2$. It then turns out that the $\mathscr{P}\mathscr{T}$-

symmetry is defined up to a phase and is given by

$$\mathscr{P}\mathscr{T}: \begin{pmatrix} \phi_1(t,\vec{x}) \\ \phi_2(t,\vec{x}) \end{pmatrix} \to \begin{pmatrix} \phi_1'(t,\vec{x}) \\ \phi_2'(t,\vec{x}) \end{pmatrix} = e^{i\gamma} \begin{pmatrix} \phi_1^*(t,\vec{x}) \\ -\phi_2^*(t,\vec{x}) \end{pmatrix} \ , \ \gamma \in \mathbb{R} \ . \tag{3.7}$$

The eigenvalues of the mass squared matrix M^2 are given by

$$M_\pm^2 = \frac{1}{2}\left(m_1^2 + m_2^2 \pm \sqrt{\left(m_1^2 - m_2^2\right)^2 - 4\mu^4}\right) \ . \tag{3.8}$$

These values are real as long as

$$\frac{|m_1^2 - m_2^2|}{2} \leq |\mu^2| \ . \tag{3.9}$$

As long as condition (3.9) is satisfied, we are in the unbroken $\mathscr{P}\mathscr{T}$-symmetric regime. Inside this regime we can look into two limiting cases. Firstly, when $\mu^2 \to 0$ the system is in the Hermitian limit. The eigenmasses squared are then simply given by m_1^2, m_2^2. Secondly, we can look into the limit

$$|\mu^2| \to \frac{|m_1^2 - m_2^2|}{2} \ . \tag{3.10}$$

In this case the two masses both become equal and are given by $M_\pm^2 = \frac{m_1^2+m_2^2}{2}$. Inside this limit the system becomes degenerate and we loose one degree of freedom. We will discuss what happens at this limit later on in this section when we discuss the eigenstates of this system.

3.1.2 Equations of Motion

For a Hermitian Lagrangian $\mathscr{L}_H$, with action $S_H = \int d^4x \, \mathscr{L}_H$, consisting of n complex scalar fields ϕ_i with $i \in \{1, 2, \cdots n\}$, the equations of motion are given by

$$0 \equiv \frac{\delta S_H}{\delta \phi_i} = \frac{\partial S_H}{\partial \phi_i} - \partial_\alpha \left(\frac{\partial S_H}{\partial\left(\partial_\alpha \phi_i\right)}\right) \quad , i \in 1, 2, \ldots n \tag{3.11a}$$

$$0 \equiv \frac{\delta S_H}{\delta \phi_i^*} = \frac{\partial S_H}{\partial \phi_i^*} - \partial_\alpha \left(\frac{\partial S_H}{\partial\left(\partial_\alpha \phi_i^*\right)}\right) \quad , i \in 1, 2, \ldots n \ . \tag{3.11b}$$

Initially one might think these $2n$ equations overconstrain the system so that only a non-dynamical solution will exist. However, since the action we consider is Hermitian, $S_H^* = S_H$, the equations of motion (3.11b) are implied by the equations of motion (3.11a),

$$0 = \left(\frac{\delta S_H}{\delta \phi_i}\right)^* = \left(\frac{\delta S_H^*}{\delta \phi_i^*}\right) = \left(\frac{S_H}{\delta \phi_i^*}\right) . \tag{3.12}$$

Because of the Hermitian nature of this system, the only restriction one needs stems from equation (3.11a) or equivalently equation (3.11b). This means that we only have n equations that constrain the fields, which do allows for dynamical solutions of our equations of motion.

Things become more complicated when the system that we consider is no longer Hermitian. In this case, the relation (3.12) is no longer satisfied and the only possible solution to the equations of motion would be the trivial one

$$\phi_i = \phi_i^* = 0 . \tag{3.13}$$

We can show the $\mathscr{P}\mathscr{T}$-symmetry of our system explicitly by writing the Lagrangian (3.1) in terms of $\mathscr{P}\mathscr{T}$-conjugate fields. We use the notation

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \text{and } \Phi^\ddagger \equiv [\mathscr{P}\mathscr{T}(\Phi)]^T = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi^*\right]^T = \begin{pmatrix} \phi_1^* & -\phi_2^* \end{pmatrix} , \tag{3.14}$$

so the Lagrangian (3.1) can be writen as

$$\mathscr{L}_s = \Phi^\ddagger \begin{pmatrix} -\Box - m_1^2 & -\mu^2 \\ -\mu^2 & \Box + m_2^2 \end{pmatrix} \Phi . \tag{3.15}$$

To deal with the previously mentioned problems concerning the equations of motion, we propose a method introduced in [69, 70]. Since both equations in (3.11) do not allow for dynamical solutions, we should choose only one of the two equations

$$0 \equiv \frac{\delta S}{\delta \Phi}, \quad \text{or} \quad 0 \equiv \frac{\delta S}{\delta \Phi^\ddagger} , \tag{3.16}$$

where $S = \int d^4x\, \mathscr{L}_s$ is the action. The other pysical equations are then obtained by complex conjugating the equations of motion choosen in (3.16)

$$0 \equiv \left(\frac{\delta S}{\delta \Phi}\right)^*, \quad \text{or} \quad 0 \equiv \left(\frac{\delta S}{\delta \Phi^\ddagger}\right)^* . \tag{3.17}$$

In short, the equations of motion for our system are either

$$0 \equiv \frac{\delta S}{\delta \Phi^\ddagger} = \left(\frac{\delta S}{\delta \Phi}\right)^\ddagger = \begin{pmatrix} -\Box\phi_1 - m_1^2\phi_1 - \mu^2\phi_2 \\ \Box\phi_2 + m_2^2\phi_2 - \mu^2\phi_1 \end{pmatrix}, \tag{3.18a}$$

$$0 \equiv \left(\frac{\delta S}{\delta \Phi^\ddagger}\right)^* = \begin{pmatrix} -\Box\phi_1^* - m_1^2\phi_1^* - \mu^2\phi_2^* \\ \Box\phi_2^* + m_2^2\phi_2^* - \mu^2\phi_1^* \end{pmatrix} \tag{3.18b}$$

or

$$0 \equiv \frac{\delta S}{\delta \Phi} = \left(\frac{\delta S}{\delta \Phi^\ddagger}\right)^\ddagger = \begin{pmatrix} -\Box\phi_1^* - m_1^2\phi_1^* + \mu^2\phi_2^* \\ -\Box\phi_2^* - m_2^2\phi_2^* - \mu^2\phi_1^* \end{pmatrix}^T, \tag{3.19a}$$

$$0 \equiv \left(\frac{\delta S}{\delta \Phi}\right)^* = \begin{pmatrix} -\Box\phi_1 - m_1^2\phi_1 + \mu^2\phi_2 \\ -\Box\phi_2 - m_2^2\phi_2 - \mu^2\phi_1 \end{pmatrix}^T . \tag{3.19b}$$

These two choices only differ in the sign of the μ^2 coupling. The physical observables (3.8) do not depend on the sign of μ^2, however, since these only depend on μ^4. Both choices of equations of motion will lead to the same observables.

An alternative approach to this problem was proposed by P. Mannheim [58]. The author proposes to transform the non-Hermitian Lagrangian (3.1) into a Hermitian one via a similarity transformation. This newly transformed Lagrangian is now Hermitian, so there should not be any problem defining the equations of motion in this frame of reference. In this work, we will focus mainly on the first approach to the equations of motion, but we will sometimes compare the results of the two different methods.

We will choose the equations of motion to be

$$\begin{cases} 0 = \Box\phi_1 + m_1^2\phi_1 + \mu^2\phi_2 \\ 0 = \Box\phi_2 + m_2^2\phi_2 - \mu^2\phi_1 \end{cases} . \tag{3.20}$$

Using these equations of motion, one can check that the right eigenfields of our system are given by

$$\mathcal{N}^{-1/2}\left(\left[m_1^2 - m_2^2 + \sqrt{\left(m_1^2 - m_2^2\right)^2 - 4\mu^4}\right]\phi_1 + 2\mu^2\phi_2\right) , \tag{3.21a}$$

with eigenmasses squared: M_+^2 , and

$$\mathcal{N}^{-1/2}\left(\left[m_1^2 - m_2^2 + \sqrt{\left(m_1^2 - m_2^2\right)^2 - 4\mu^4}\right]\phi_2 + 2\mu^2\phi_1\right) , \tag{3.21b}$$

with eigenmasses squared: M_-^2 ,

where

$$\mathcal{N} = 2\sqrt{\left(m_1^2 - m_2^2\right)^2 - 4\mu^4}\left|\left(m_1^2 - m_2^2\right) + \sqrt{\left(m_1^2 - m_2^2\right)^2 - 4\mu^4}\right| . \tag{3.22}$$

The left eigenfields are given by

$$\theta\left(m_1^2 - m_2^2\right)\mathcal{N}^{-1/2}\left(\left[m_1^2 - m_2^2 + \sqrt{\left(m_1^2 - m_2^2\right)^2 - 4\mu^4}\right]\phi_1^* - 2\mu^2\phi_2^*\right) , \tag{3.23a}$$

with eigenmasses squared: M_+^2 , and

$$\theta\left(m_1^2 - m_2^2\right)\mathcal{N}^{-1/2}\left(\left[m_1^2 - m_2^2 + \sqrt{\left(m_1^2 - m_2^2\right)^2 - 4\mu^4}\right]\phi_2^* - 2\mu^2\phi_1^*\right) , \tag{3.23b}$$

with eigenmasses squared: M_-^2 .

Remark that the left and right eigenvectors are related by a $\mathscr{C}'\mathscr{P}\mathscr{T}$ transformation so that they form an orthonormal basis under the $\mathscr{C}'\mathscr{P}\mathscr{T}$ inner product. Remark that the overal sign of the $\mathscr{C}'$-operator depends on the relative values of the m_1^2 and m_2^2 couplings, but is independent of the non-Hermitian coupling μ^2.

It proves interesting to look at how these vectors depend on the coupling μ^2. To

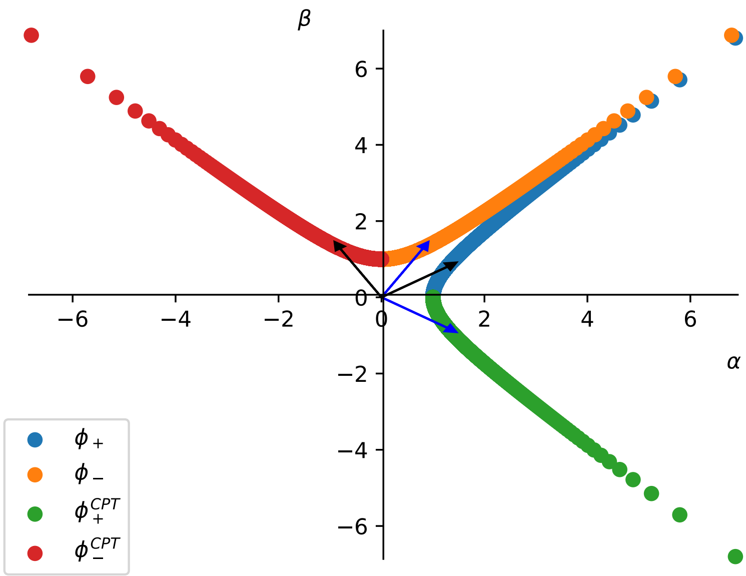

Figure 3.1: We plot the component of the left and right eigenvectors for the different eigenvectors. We see that when $\mu^2 = 0$, the components are $(1,0)$ and $(0,1)$. When $\mu^2 \to \frac{|m_1^2 - m_2^2|}{2}$, we can see that $\alpha_\pm^{(1)} = \beta_\pm^{(1)}$ for both right eigenvectors so that they become parallel. In this limit, the right eigenvectors become equal to each other and the left eigenvectos become negative towards each other.

examine this behaviour, we write the right eigenfields as

$$\phi_\pm = \alpha_\pm^{(1)} \phi_1 + \beta_\pm^{(1)} \phi_2 \ , \tag{3.24}$$

and the left eigenfields as

$$\phi_\pm^{\mathscr{C}'\mathscr{P}\mathscr{T}} = \alpha_\pm^{(2)} \phi_1^\dagger + \beta_\pm^{(2)} \phi_2^\dagger \ . \tag{3.25}$$

We have plotted the components $\alpha_\pm^{(1)}$, $\alpha_\pm^{(2)}$, $\beta_\pm^{(1)}$ and $\beta_\pm^{(2)}$ for different values of μ^2 in figure 3.1. In this plot in figure 3.1, the arrows indicate the components of $\phi_+, \phi_-, \phi_+^{\mathscr{C}'\mathscr{P}\mathscr{T}}, \phi_-^{\mathscr{C}'\mathscr{P}\mathscr{T}}$ for $\mu^2 = 0.9 * \frac{|m_1^2 - m_2^2|}{2}$. Remark that the components of $\phi_\pm$ and $\phi_\mp^{\mathscr{C}'\mathscr{P}\mathscr{T}}$ always appear under an angle of $\frac{\pi}{2}$. This confirms the orthogonality of the eigenvectors under the $\mathscr{C}'\mathscr{P}\mathscr{T}$ inner product.

For the case $\mu^2 = 0$, the components of ϕ_- are equal to those of $\phi_-^{\mathscr{C}'\mathscr{P}\mathscr{T}}$ and the components of ϕ_+ are equal to those of $\phi_+^{\mathscr{C}'\mathscr{P}\mathscr{T}}$. The components of ϕ_+ and ϕ_- form an angle of $\frac{\pi}{2}$ in that limit. For increasing μ^2, the angle between ϕ_+ and ϕ_- decreases until it finally is zero when $\mu^2 \to \frac{|m_1^2-m_2^2|}{2}$ and $\phi_+ = \phi_-$. In this limit the system looses two degrees of freedom.

3.1.3 Symmetries and Conserved Currents

As mentioned before, for a Hermitian system that possesses a continuous symmetry there is a current that is conserved. When $\mu^2 \to 0$, we are in such a Hermitian limit and the system is indeed conserved under two $U(1)$ transformations of the form

$$\varphi_1 : \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} e^{-i\gamma_1}\phi_1 \\ \phi_2 \end{pmatrix} \text{ and } \varphi_2 : \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \phi_1 \\ e^{-i\gamma_2}\phi_2 \end{pmatrix} , \tag{3.26}$$

with $\gamma_1, \gamma_2 \in \mathbb{R}$. The corresponding Noether currents to these transformations are

$$j_1^\alpha = i\left(\phi_1^* \partial^\alpha \phi_1 - \phi_1 \partial^\alpha \phi_1^*\right) \tag{3.27}$$

$$j_2^\alpha = i\left(\phi_2^* \partial^\alpha \phi_2 - \phi_2 \partial^\alpha \phi_2^*\right) . \tag{3.28}$$

When $\mu^2 \neq 0$, these currents are no longer individually conserved, but instead

$$\partial_\alpha j_1^\alpha = \partial_\alpha j_2^\alpha = i\mu^2\left(\phi_2^*\phi_1 - \phi_1^*\phi_2\right) . \tag{3.29}$$

In the non-Hermitian case, $\mu^2 \neq 0$, the system (3.1) is only invariant under one $U(1)$ transformation of the form

$$\varphi : \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to e^{-i\gamma} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} , \quad \text{with } \gamma \in \mathbb{R} . \tag{3.30}$$

From equation (3.29) it is clear that the Noether current corresponding to this non-Hermitian transformation (3.30)

$$j'^\alpha = j_1^\alpha + j_2^\alpha = i\left(\left[\phi_1^*\partial^\alpha\phi_1 - \phi_1\partial^\alpha\phi_1^*\right] + \left[\phi_2^*\partial^\alpha\phi_2 - \phi_2\partial^\alpha\phi_2^*\right]\right) , \tag{3.31}$$

is no longer conserved

$$\partial_\alpha j'^\alpha = \partial_\alpha j_1^\alpha + \partial_\alpha j_2^\alpha = 2i\mu^2 \left(\phi_2^* \phi_1 - \phi_1^* \phi_2\right) \neq 0 \; . \tag{3.32}$$

However, the current $j^\alpha = j_1^\alpha - j_2^\alpha$ is conserved under the equations of motion. To understand this remarkable feature we need to reconsider the derivation of the Noether current (2.19) that we gave previously in chapter (2.1.2).

We have a transformation φ', under which the fields change as

$$\Phi \to \Phi + \delta\Phi \quad , \quad \Phi^\dagger \to \Phi^\dagger + \delta\Phi^\dagger \; . \tag{3.33}$$

The variation of the Lagrangian $\mathscr{L}$ under such a transformation is given by

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\Phi} \cdot \delta\Phi + \delta\Phi^\dagger \cdot \frac{\partial\mathscr{L}}{\partial\Phi^\dagger} + \frac{\partial\mathscr{L}}{\partial\left(\partial_\alpha\Phi\right)} \cdot \delta\left(\partial_\alpha\Phi\right) + \delta\left(\partial_\alpha\Phi^\dagger\right) \cdot \frac{\partial\mathscr{L}}{\partial\left(\partial_\alpha\Phi^\dagger\right)} \tag{3.34}$$

$$= \left(\frac{\partial\mathscr{L}}{\partial\Phi} - \partial_\alpha \frac{\partial\mathscr{L}}{\partial\left(\partial_\alpha\Phi\right)}\right) \cdot \delta\Phi + \delta\Phi^\dagger \cdot \left(\frac{\partial\mathscr{L}}{\partial\Phi^\dagger} - \partial_\alpha \frac{\partial\mathscr{L}}{\partial\left(\partial_\alpha\Phi^\dagger\right)}\right) \tag{3.35}$$

$$+ \partial_\alpha \left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\alpha\Phi\right)} \cdot \delta\Phi + \delta\Phi^\dagger \cdot \frac{\partial\mathscr{L}}{\partial\left(\partial_\alpha\Phi^\dagger\right)}\right)$$

$$= \left(\frac{\partial\mathscr{L}}{\partial\Phi} - \partial_\alpha \frac{\partial\mathscr{L}}{\partial\left(\partial_\alpha\Phi\right)}\right) \cdot \delta\Phi + \delta\Phi^\dagger \cdot \left(\frac{\partial\mathscr{L}}{\partial\Phi^\dagger} - \partial_\alpha \frac{\partial\mathscr{L}}{\partial\left(\partial_\alpha\Phi^\dagger\right)}\right) + \partial_\alpha \delta j^\alpha \; , \tag{3.36}$$

where

$$\delta j^\alpha = \frac{\partial\mathscr{L}}{\partial\left(\partial_\alpha\Phi\right)} \delta\Phi + \delta\Phi^\dagger \frac{\partial\mathscr{L}}{\partial\left(\partial_\alpha\Phi^\dagger\right)} \; . \tag{3.37}$$

Remark that if the transformation is a symmetry, then $\delta\mathscr{L} = 0$. The two equations

$$\left(\frac{\partial\mathscr{L}}{\partial\Phi} - \partial_\alpha \frac{\mathscr{L}}{\partial\left(\partial_\alpha\Phi\right)}\right) \quad \text{and} \quad \left(\frac{\partial\mathscr{L}}{\partial\Phi^\dagger} - \partial_\alpha \frac{\mathscr{L}}{\partial\left(\partial_\alpha\Phi^\dagger\right)}\right) \; , \tag{3.38}$$

cannot simultaniously be zero if $\mathscr{L}$ is non-Hermitian. Thus $\partial_\alpha j^\alpha$ cannot be zero either and δj^α will not be conserved. Conversely, if δj^α is conserved and the equations of motions are choosen such that

$$\left(\frac{\delta \mathscr{L}}{\partial \Phi^\dagger} - \partial_\alpha \frac{\partial \mathscr{L}}{\partial\left(\partial_\alpha \Phi^\dagger\right)}\right) = 0 \,, \tag{3.39}$$

then the corresponding transformation should change the Lagrangian such that

$$\delta \mathscr{L} = \left(\frac{\partial \mathscr{L}}{\partial \Phi} - \partial_\alpha \frac{\partial \mathscr{L}}{\partial\left(\partial_\alpha \Phi\right)}\right) \cdot \delta\Phi \,. \tag{3.40}$$

For our non-Hermitian system (3.1) and the transformation (3.30), we find that

$$\left(\frac{\partial \mathscr{L}_s}{\partial \Phi} - \partial_\alpha \frac{\partial \mathscr{L}_s}{\partial\left(\partial_\alpha \Phi\right)}\right) \cdot \delta\Phi = 2i\mu^2\gamma(\phi_1^*\phi_2 - \phi_2^*\phi_1) \,, \tag{3.41}$$

which confirms that

$$\partial_\alpha j'^\alpha = 2i\mu^2\left(\phi_1^*\phi_2 - \phi_2^*\phi_1\right) \,. \tag{3.42}$$

If we look at the transformation

$$\varphi : \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} e^{-i\gamma}\phi_1 \\ e^{+i\gamma}\phi_2 \end{pmatrix} \,, \quad \text{with } \gamma \in \mathbb{R} \,, \tag{3.43}$$

then we can see that under this transformation

$$\delta \mathscr{L}_s = 2i\gamma\mu^2\left(\phi_1^*\phi_2 + \phi_2^*\phi_1\right) = \left(\frac{\partial \mathscr{L}_s}{\partial \Phi} - \partial_\alpha \frac{\partial \mathscr{L}_s}{\partial\left(\partial_\alpha \Phi\right)}\right) \cdot \delta\Phi \,, \tag{3.44}$$

and thus the current

$$j^\alpha = j_1^\alpha - j_2^\alpha \,, \tag{3.45}$$

is conserved. Remark that under this transformation with conserved current, the two complex fields possess opposite charges. Therefore, one acts as a source and another as a sink. Remark that this reflects our discussion on sink and sources in $\mathscr{P}\mathscr{T}$-symmetric theories.

For this scalar model we have shown that there are two important transformations. Firstly, there is a $U(1)$ transformation of the form

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to e^{-i\gamma}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \quad , \quad \text{with } \gamma \in \mathbb{R} \ , \tag{3.46}$$

which is a symmetry of our system. All observables are trivially invariant under such a transformation. Unlike in the Hermitian case, the corresponding current is not conserved.

The second transformation of importance is given by

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}e^{-i\gamma}\phi_1\\ e^{i\gamma}\phi_2\end{pmatrix} \quad , \quad \text{with } \gamma \in \mathbb{R} \ , \tag{3.47}$$

which is not a symmetry of the Lagrangian. The corresponding current of this transformation is given by

$$j^{\alpha} = i\left([\phi_1^*\partial^{\alpha}\phi_1 - \phi_1\partial^{\alpha}\phi_1^*] - [\phi_2^*\partial^{\alpha}\phi_2 - \phi_2\partial^{\alpha}\phi_2^*]\right) \ , \tag{3.48}$$

and is conserved. This transformation transforms the Lagrangian into

$$\mathscr{L}' = \partial_{\alpha}\phi_1^*\partial^{\alpha}\phi_1 + \partial_{\alpha}\phi_2^*\partial^{\alpha}\phi_2 - m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2 e^{2i\gamma}\phi_1^*\phi_2 + \mu^2 e^{-2i\gamma}\phi_1\phi_2^* \ . \tag{3.49}$$

Even though this Lagrangian, $\mathscr{L}'$, is different from the original Lagrangian, the spectrum of $\mathscr{L}'$ is identical to the spectrum of $\mathscr{L}$. So whilst the Lagrangian $\mathscr{L}$ is not invariant under this transformation, physical quantities such as eigenmasses are. This shows that we can interpret (3.49) as a one-parameter-family of physically equivalant Lagrangians. This feature can serve as a generalisation of the choice we have in defining our equations of motion. A different choice of our equations of motion would correspond to the original equations of motion of a transformed Lagrangian under the transformation (3.46) with $\varepsilon = \frac{\pi}{2}$.

Non-Hermitian theories are different from Hermitian ones in that currents corresponding to a symmetry are no longer conserved. Instead, it is the current corresponding to a transformation for which

$$\delta\mathscr{L} = \left(\frac{\partial\mathscr{L}}{\partial\Phi} - \partial_\alpha \frac{\mathscr{L}}{\partial\left(\partial_\alpha\right)}\right)\cdot\delta\Phi\ , \tag{3.50}$$

that will be conserved. Remark that in the Hermitian limit both

$$\frac{\partial\mathscr{L}}{\partial\Phi} - \partial_\alpha \frac{\partial\mathscr{L}}{\partial\left(\partial_\alpha\Phi\right)} = 0 \quad \text{and} \quad \frac{\partial\mathscr{L}}{\partial\Phi^\dagger} - \partial_\alpha \frac{\partial\mathscr{L}}{\partial\left(\partial_\alpha\Phi^\dagger\right)} = 0\ , \tag{3.51}$$

so equation (3.50) still holds in this case.

3.2 Non-Hermitian Fermionic Model

3.2.1 Equations of Motion and Discrete Symmetries

In the previous part, we discussed a non-Hermitian scalar field Lagrangian. We noted two main differences compared to Hermitian systems. The first difference lies in the definition of the equations of motion, and the second difference lies in the correspondence between transformations and conserved currents. Remark that the derivations for both of these results should work for a general non-Hermitian Lagrangian. The same results should thus hold for other non-Hermitian Lagrangians then the one in (3.1). Here we explore a non-Hermitian Fermionic Fermionic Lagrangian of the form

$$\mathscr{L}_F = \overline{\psi}\left(i\partial\!\!\!/ - m - \mu_f\gamma^5\right)\psi\ , \text{ where } \overline{\psi} = \psi^\dagger\gamma^0\ . \tag{3.52}$$

This model has been studied in works such as [45, 49, 50, 69, 70].

The equations of motion for this system are chosen to be

$$0 \equiv \frac{\partial\mathscr{L}_F}{\partial\overline{\psi}} - \partial_\alpha \frac{\partial\mathscr{L}_F}{\partial\left(\partial_\alpha\overline{\psi}\right)} = \left(i\partial\!\!\!/ - m - \mu_f\gamma^5\right)\psi \tag{3.53}$$

and

$$0 \equiv \left(\frac{\partial \mathscr{L}_F}{\partial \overline{\psi}} - \partial_\alpha \frac{\partial \mathscr{L}_F}{\partial \left(\partial_\alpha \overline{\psi} \right)} \right)^\dagger \gamma^0 = \overline{\psi} \left(-i \overleftarrow{\not{\partial}} - m + \mu_f \gamma^5 \right) \ . \tag{3.54}$$

From these equations of motion one can see that

$$\left(-i\not{\partial} - m + \mu_f \gamma^5 \right) \left(i\not{\partial} - m - \mu_f \gamma^5 \right) \psi = \left(\Box + \left[m^2 - \mu_f^2 \right] \right) \psi = 0 \ . \tag{3.55}$$

Thus the physical mass squared of this sytem is $m^2 - \mu_f^2$, which means that the eigenmasses are real as long as the condition

$$m^2 \geq \mu_f^2 \ , \tag{3.56}$$

is satisfied. Remark that we could have choosen the equations of motion to be

$$0 \equiv \partial_\alpha \frac{\partial \mathscr{L}_F}{\partial \left(\partial \psi \right)} - \frac{\partial \mathscr{L}_F}{\partial \psi} = \overline{\psi} \left(\overleftarrow{\not{\partial}} + m + \mu_f \gamma^5 \right) \tag{3.57}$$

and

$$0 \equiv -\gamma^0 \left(\partial_\alpha \frac{\mathscr{L}_F}{\partial \left(\partial_\alpha \psi \right)} - \frac{\partial \mathscr{L}_F}{\partial \psi} \right)^\dagger = \left(i\not{\partial} - m + \mu_f \gamma^5 \right) \psi \ . \tag{3.58}$$

The difference with these equations of motion, compared to the equations of motion as defined in (3.53), is in the sign of μ_f. This difference in sign does not change the eigenmasses squared, since these only depend on μ_f^2.

The discrete symmetries of this system are defined as

$$\mathscr{P} : \psi(\vec{x},t) \to \gamma^0 \psi(\vec{x},t) \tag{3.59a}$$

$$\mathscr{P} : \overline{\psi}(\vec{x},t) \to \overline{\psi}(\vec{x},t) \gamma^0 \tag{3.59b}$$

and

$$\mathscr{T} : \psi(\vec{x},t) \to i\gamma^1 \gamma^2 \psi^*(\vec{x},t) \tag{3.60a}$$

$$\mathscr{T} : \overline{\psi}(\vec{x},t) \to \overline{\psi}^*(\vec{x},t) i\gamma^1 \gamma^2 \ . \tag{3.60b}$$

This shows that the Lagrangian (3.52) is $\mathscr{P}\mathscr{T}$-symmetric. From here we can see that as long as the condition (3.56) is satisfied, the masses are real and this system is in the unbroken $\mathscr{P}\mathscr{T}$-symmetric regime.

3.2.2 Continuous Transformations and Conserved Currents

Let us assume we have a transformation that acts on the fields as

$$\psi \to \psi + \delta\psi \quad , \quad \overline{\psi} \to \overline{\psi} + \delta\overline{\psi} \, . \tag{3.61}$$

The Lagrangian changes under this transformation as

$$\begin{aligned}
\delta\mathscr{L}_F &= \frac{\partial\mathscr{L}_F}{\partial\psi}\delta\psi + \delta\overline{\psi}\frac{\partial\mathscr{L}_F}{\partial\overline{\psi}} + \frac{\partial\mathscr{L}_F}{\partial\left(\partial_\alpha\psi\right)}\partial_\alpha\left(\delta\psi\right) + \partial_\alpha\left(\delta\overline{\psi}\right)\frac{\partial\mathscr{L}_F}{\partial\left(\partial_\alpha\overline{\psi}\right)} \qquad (3.62)\\
&= \left(\frac{\partial\mathscr{L}_F}{\partial\psi} - \partial_\alpha\frac{\partial\mathscr{L}_F}{\partial\left(\partial_\alpha\psi\right)}\right)\delta\psi + \delta\overline{\psi}\left(\frac{\partial\mathscr{L}_F}{\partial\overline{\psi}} - \partial_\alpha\frac{\partial\mathscr{L}_F}{\partial\left(\partial_\alpha\overline{\psi}\right)}\right) + \partial_\alpha\left(\delta j_f^\alpha\right) \, ,
\end{aligned}$$

where the current is given by

$$\delta j_f^\alpha = \frac{\partial\mathscr{L}_F}{\partial\left(\partial_\alpha\psi\right)}\delta\psi + \delta\overline{\psi}\frac{\partial\mathscr{L}_F}{\partial\left(\partial_\alpha\overline{\psi}\right)} = \frac{i}{2}\left(\overline{\psi}\gamma^\alpha\delta\psi - \delta\overline{\psi}\gamma^\alpha\psi\right) \, . \tag{3.63}$$

This current will only be conserved under a transformation for which

$$\delta\mathscr{L}_F = \left(\frac{\partial\mathscr{L}_F}{\partial\psi} - \partial_\alpha\frac{\partial\mathscr{L}_F}{\partial\left(\partial_\alpha\psi\right)}\right)\delta\psi = -2\mu_f\overline{\psi}\gamma^5\delta\psi \, . \tag{3.64}$$

Such a transformation is given by

$$\psi \to \psi' = \exp\left[+i\alpha\left(1 + \frac{\mu_f}{m}\gamma^5\right)\right]\psi \tag{3.65}$$

$$\bar{\psi} \to \overline{\psi}' = \overline{\psi}\exp\left[-i\alpha\left(1 - \frac{\mu_f}{m}\gamma^5\right)\right] \, , \tag{3.66}$$

and thus its corresponding current

$$\delta j_f^\alpha = \overline{\psi}\gamma^\alpha\left(1 + \frac{\mu_f}{m}\gamma^5\right)\psi \, , \tag{3.67}$$

is conserved.

3.3 Conclusion

In this section we have studied two non-Hermitian Quantum Field Theories. Firstly, we discussed the scalar model

$$\mathcal{L}_s = \begin{pmatrix} \partial_\alpha \phi_1^* & \partial_\alpha \phi_2^* \end{pmatrix} \begin{pmatrix} \partial^\alpha \phi_1 \\ \partial^\alpha \phi_2 \end{pmatrix} - \begin{pmatrix} \phi_1^* & \phi_2^* \end{pmatrix} \begin{pmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} , \tag{3.68}$$

with anti-Hermitian mass term $\mu^2 (\phi_1^* \phi_2 - \phi_2^* \phi_1)$. Secondly, we discussed a fermionic model

$$\mathcal{L}_F = \overline{\psi} \left(i\partial\!\!\!/ - m - \mu_f \gamma^5 \right) \psi , \tag{3.69}$$

with the non-Hermitian term $\mu_f \overline{\psi} \gamma^5 \psi$. We discussed the discrete $\mathscr{P}$ and $\mathscr{T}$ transformations for both these models and show that they are $\mathscr{P}\mathscr{T}$-symmetric. The spectrum of both these models is real for a certain region in parameter space given by

$$\frac{|m_1^2 - m_2^2|}{2} \leq |\mu^2| , \tag{3.70}$$

for the scalar model and

$$m^2 \geq \mu_f^2 , \tag{3.71}$$

for the fermionic model.

Special care needs to be taken when defining the equation of motion for such systems. For these non-Hermitian systems we need to define the equations of motion by choosing one of the following equations for the scalar fields

$$\frac{\partial \mathcal{L}_s}{\partial \phi_i^*} = 0 \quad \text{or} \quad \frac{\partial \mathcal{L}_s}{\partial \phi_i} = 0 , \tag{3.72}$$

and for the Fermionic fields

$$\frac{\partial \mathcal{L}_F}{\partial \overline{\psi}} = 0 \quad \text{or} \quad \frac{\partial \mathcal{L}_F}{\partial \psi_i} = 0 . \tag{3.73}$$

When the anti-Hermitian terms are not zero, both these equations cannot be satisfied simultaneously. As a consequence we have shown that the currents

$$j^{\alpha} = \frac{\partial \mathscr{L}}{\partial\left(\partial_{\alpha}\phi_i\right)}\delta\phi_i + \frac{\partial \mathscr{L}}{\partial\left(\partial_{\alpha}\phi_i^*\right)}\delta\phi_i^* \tag{3.74}$$

and

$$j_f^{\alpha} = \frac{\partial \mathscr{L}_f}{\partial\left(\partial_{\alpha}\psi\right)}\delta\psi + \delta\overline{\psi}\frac{\partial \mathscr{L}}{\partial\left(\partial_{\alpha}\overline{\psi}\right)} \tag{3.75}$$

can only be conserved if the corresponding current is not a symmetry, but instead a transformation for which

$$\delta\mathscr{L}_s = \left(\frac{\partial \mathscr{L}_s}{\partial\phi_i} - \partial_{\alpha}\frac{\partial \mathscr{L}_s}{\partial\left(\partial_{\alpha}\phi_i\right)}\right)\delta\phi_i + \left(\frac{\partial \mathscr{L}_s}{\partial\phi_i^*} - \partial_{\alpha}\frac{\partial \mathscr{L}_s}{\partial\left(\partial_{\alpha}\phi_i^*\right)}\right)\delta\phi_i^* \ , \tag{3.76}$$

or

$$\delta\mathscr{L}_F = \left(\frac{\partial \mathscr{L}_F}{\partial\psi} - \partial_{\alpha}\frac{\partial \mathscr{L}_F}{\partial\left(\partial_{\alpha}\psi\right)}\right)\delta\psi + \delta\overline{\psi}\left(\frac{\partial \mathscr{L}_F}{\partial\overline{\psi}} - \partial_{\alpha}\frac{\partial \mathscr{L}_F}{\partial\left(\partial_{\alpha}\overline{\psi}\right)}\right) \ . \tag{3.77}$$

Such a transformation is given by

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \exp\left[i\gamma_s\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \text{ ,with } \gamma_s \in \mathbb{R} \ , \tag{3.78}$$

for the scalar Lagrangian with conserved current

$$j_s^{\alpha} = i\left[\left(\partial^{\alpha}\phi_1^*\phi_1 - \partial^{\alpha}\phi_1\phi_1^*\right) - \left(\partial^{\alpha}\phi_2^*\phi_2 - \partial^{\alpha}\phi_2\phi_2^*\right)\right] \ , \tag{3.79}$$

and

$$\psi \to \exp\left[i\gamma_f\left(1 + \frac{\mu_f}{m}\gamma^5\right)\right]\psi \text{ ,with } \gamma_f \in \mathbb{R} \ , \tag{3.80}$$

for the Fermionic Lagrangian with conserved current

$$j_f^{\alpha} = \bar{\psi}\gamma^{\alpha}\left(1 + \frac{\mu_f}{m}\gamma^5\right)\psi \ . \tag{3.81}$$

The transformation (3.78) transforms the Lagrangian $\mathscr{L}_s$ into a set of one parameter Lagrangians, all with the same physical observables.

Chapter 4

Goldstone Theorem and the Englert-Brout-Higgs Mechanism

In the previous section, we discussed how to consistently define the equation of motion and the relationship between transformations and conserved currents for non-Hermitian, $\mathscr{PT}$-symmetric models. It turns out that many results that hold for Hermitian Quantum Field Theory need to be reexamined when applied in a non-Hermitian context.

A straightforward next step in the study of non-Hermitian models like (3.1) is to examine whether such models can produce results similar to those predicted by the Standard Model. In this section, we will check whether the mechanism of mass generation for a gauge field as described by the Englert-Brout-Higgs mechanism ([56], [57]) still works for non-Hermitian models like (3.1).

With this in mind, we first need to make sure if and how the Goldstone theorem [71] holds for a general non-Hermitian model. It turns out that for a Nambu-Goldstone boson to exist, we will need a non-trivial vacuum expectation value that is broken by a transformation with a conserved current. As we have seen in the previous section, such a transformation will, in general, not be a symmetry of the system. We will discuss the derivation of the Nambu-Goldstone theorem for a non-Hermitian model and explicitly show the spectrum for a system like (3.1). We follow the outline as described in [72].

After this, we want to gauge our scalar non-Hermitian model and replace the usual

derivatives by covariant ones. We examine two particular models that are both a straightforward generalisation of our scalar model. In the first model the gauge field couples to a conserved current. We will see that this model cannot be physical since it has a non-zero polarisation tensor. The second model couples the gauge field to a non-conserved current. To still have physically consistent equations of motion, we will see that a gauge restriction must be imposed on the level of the Lagrangian. This will closely follow the work done in [72].
The final part of this section will highlight an alternative approach to these problems. This starts from the alternative approach to the equations of motion as suggested in [58] and already discussed in section (3.1.2). We are then able to compare the two results that were obtained using the different equations of motion.

4.1 Goldstone Theorem

In the previous section, we discussed continuous global transformations in the context of non-Hermitian Quantum Field Theories and how they relate to conserved currents. This relation differs from the usual ones in Hermitian theories. It might thus be interesting to investigate whether other results still hold for non-Hermitian theories. The first result we want to discuss is whether the Goldstone theorem still holds for our non-Hermitian scalar model. To do this, we must first discuss spontaneous symmetry breaking.

4.1.1 Spontaneous Symmetry Breaking

We expand the model that was introduced in (3.1) by including an interacting term of the form

$$\mathscr{L}_{\text{int}} = \frac{\kappa}{4}|\phi_1|^4 \,, \tag{4.1}$$

and reverse the sign of the $|\phi_1|^2$ coupling so that we end up with a Lagrangian

$$\mathscr{L} = \partial_\alpha \phi_1^* \partial^\alpha \phi_1 + \partial_\alpha \phi_2^* \partial^\alpha \phi_2 + m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2\left(\phi_1^*\phi_2 - \phi_2^*\phi_1\right) - \frac{\kappa}{4}|\phi_1|^4 \,. \tag{4.2}$$

This is done so that this system can possesses a non-trivial vacuum expectation value. This vacuum expectation value is given by the solution of the equations

$$\frac{\delta U}{\delta \phi_1^*} = \frac{\kappa}{4}|\phi_1|^2\phi_1 - m_1^2\phi_1 + \mu^2\phi_2 = 0 \,, \tag{4.3a}$$

$$\frac{\delta U}{\delta \phi_2^*} = m_2^2\phi_2 - \mu^2\phi_1 = 0 \,, \tag{4.3b}$$

where U is the potential of the Lagrangian (4.2) and in this case given by

$$U = -m_1^2|\phi_1|^2 + m_2^2|\phi_2|^2 + \mu^2\left(\phi_1^*\phi_2 - \phi_2^*\phi_1\right) + \frac{\kappa}{4}|\phi_1|^4 \,. \tag{4.4}$$

Remark that these solutions can also be seen as the non-dynamical solutions to the equations of motion. Within the definition of the vacuum expectation value (4.3), a choice in equations of motion is assumed.
Similar to the Lagrangian (3.1), the Lagrangian (4.2) is also invariant under a global transformation of the form

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to e^{-i\gamma}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \text{ ,with } \quad \gamma \in \mathbb{R} \,. \tag{4.5}$$

The solutions to equations (4.3) are also all connected to each other by these transformations. These solutions are given by

$$\begin{pmatrix}v_1\\ v_2\end{pmatrix} \equiv \sqrt{\frac{2}{\kappa}\left(m_1^2 - \frac{\mu^4}{m_2^2}\right)}\begin{pmatrix}1\\ \mu^2/m_2^2\end{pmatrix} e^{-i\varepsilon} \text{ ,with } \quad \varepsilon \in \mathbb{R} \,. \tag{4.6}$$

We choose a particular vacuum by setting $\varepsilon = 0$, which fixes an angle in (4.6). These non-trivial vacuum expectation values form the true vacuums of our system. The vacuum expectation value $\langle\Phi\rangle = 0$, is a false vacuum. Such a vacuum is unstable and a field at this vacuum will tend to move to a lower, more stable vacuum after which it will oscillate around this new vacuum. If we want to consider the fields as physical fluctuations around the true vacuum, we should represent them as

$$\phi_1 = v_1 + \hat{\phi}_1 \quad \text{and} \quad \phi_2 = v_2 + \hat{\phi}_2 \ , \tag{4.7}$$

where $\hat{\phi}_i$ represents those fluctations around the true vacuum. The Lagrangian (4.2) in terms of these new fields becomes

$$\begin{aligned}\mathscr{L} =& \partial_\alpha \hat{\phi}_1^* \partial^\alpha \hat{\phi}_1 + \partial_\alpha \hat{\phi}_2^* \partial^\alpha \hat{\phi}_2 + \frac{2\mu^4}{m_2^2} v_1 \hat{\phi}_1 - 2m_2^2 v_2 \hat{\phi}_2 - \tilde{m}_1^2 |\hat{\phi}_1|^2 - \frac{\kappa}{4} v_1^2 \left(\hat{\phi}_1^2 + \left(\hat{\phi}_1^*\right)^2\right) \\ & - m_2^2 |\hat{\phi}_2|^2 - \mu^2 \left(\hat{\phi}_1^* \hat{\phi}_2 - \hat{\phi}_2^* \hat{\phi}_1\right) - \frac{\kappa}{2} v_1 \left(\hat{\phi}_1^* + \hat{\phi}_1\right) |\hat{\phi}_1|^2 - \frac{\kappa}{4} |\hat{\phi}_1|^4 \ ,\end{aligned} \tag{4.8}$$

where

$$\tilde{m}_1^2 \equiv \kappa v_1^2 - m_1^2 \ . \tag{4.9}$$

The linear terms in the Lagrangian are a direct consequence of the non-Hermitian behaviour and our choice of equations of motion. These terms will not be physically important since they do not play a role in the equations of motion. Remark that unlike the Lagrangian (4.2), the Lagrangian (4.8) does depend on the choice of equations of motion, since the fields in (4.8) are fluctuations around a vacuum. To define this vacuum, one needs to choose a set of equations of motion and this choice is implicit in the formulation of (4.8).

Remark that this entire description only keeps the $U(1)$ symmetry until equation (4.7). By fixing a particular angle in (4.7), we explicitly break the $U(1)$ symmetry in our system. This process is known as a spontaneous symmetry breaking.

4.1.2 Goldstone Theorem

In Hermitian theories, a system with spontaneous symmetry breaking implies the appearance of a massless particle, known as the Nambu-Goldstone boson. This feature is known as the Goldstone theorem.

Goldstone Theorem. *For a continuous symmetry that is spontaneously broken, it follows that there exists one massless particle - called a Nambu-Goldstone boson - for every generator of the symmetry that is broken.*

The proof of this theorem, which can be found in works such as [71, 73, 74], relies on the existence of a conserved current and not on the invariance of a Lagrangian under such a transformation. Both these things are equivalent for Hermitian theories. This is however not the case for a non-Hermitian model (4.2). This Lagrangian is invariant under a transformation of the form

$$\Phi \to \exp(-i\gamma)\Phi\ , \quad \text{with } \gamma \in \mathbb{R}\ , \tag{4.10}$$

while the conserved current corresponds to the transformation

$$\Phi \to \exp(-i\gamma P)\Phi\ , \quad \text{with } \gamma \in \mathbb{R}\ , \tag{4.11}$$

where

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{4.12}$$

The latter transforms the Lagrangian (4.2) into a set of one-parameter-family of physically equivalent Lagrangians with potentials given by

$$U \to U'_\gamma = -m_1^2|\phi_1|^2 + m_2^2|\phi_2|^2 + \mu^2\left(e^{2i\gamma}\phi_1^*\phi_2 - e^{-2i\gamma}\phi_2^*\phi_1\right) + \frac{\kappa}{4}|\phi_1|^4\ . \tag{4.13}$$

It is the existence of this latter transformation that will be important for the proof of the Goldstone theorem in the non-Hermitian case. In what follows, we revisit the derivation of the Goldstone theorem in the context of such a non-Hermitian theory.

Proof. We assume that there exists an infinitesimal transformation, which takes the generic form

$$\Phi \ \to\ \Phi + i\varepsilon T\Phi\ , \quad \text{with } \ \varepsilon \in \mathbb{R}\ , \tag{4.14}$$

where T is the generator of this transformation. We also assume that this transformation corresponds to a conserved current j^ν, with conserved charge $Q = \int \mathrm{d}^3\mathbf{x}\ j^0(x)$. Most importantly, for the non-Hermitian theory, this transformation does not leave the Lagrangian invariant.

We are interested in the vacuum expectation of the commutator $[Q,\Phi(x)]$:

$$\langle\bar{0}|[Q,\Phi(x)]|0\rangle \;=\; iT\,\langle\Phi\rangle\;, \tag{4.15}$$

where $\langle\Phi\rangle \equiv \langle\bar{0}|\Phi(x)|0\rangle$. We note that the physical inner product is defined with respect to $\mathscr{C}'\mathscr{P}\mathscr{T}$, as is necessary for a non-Hermitian theory. With this exception, the proof of the Goldstone theorem proceeds in the same manner as for Hermitian theories (and we closely follow the proof given in [75]). By inserting complete sets of intermediate states, we can write

$$\begin{aligned}\langle\bar{0}|[j^\alpha(y),\Phi(x)]|0\rangle \;&=\; \sum_N\Big[\langle\bar{0}|j^\alpha(y)|N\rangle\langle\bar{N}|\Phi(x)|0\rangle \;-\; \langle\bar{0}|\Phi(x)|N\rangle\langle\bar{N}|j^\alpha(y)|0\rangle\Big]\\ &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\Bigg[e^{-ip\cdot(y-x)}\sum_N(2\pi)^4\delta^4(p_N-p)\langle\bar{0}|j^\alpha(0)|N\rangle\langle\bar{N}|\Phi(0)|0\rangle\\ &\qquad - e^{ip\cdot(y-x)}\sum_N(2\pi)^4\delta^4(p_N-p)\langle\bar{0}|\Phi(0)|N\rangle\langle\bar{N}|j^\alpha(0)|0\rangle\Bigg]\;,\end{aligned} \tag{4.16}$$

and, by virtue of Lorentz invariance, these terms should be of the form

$$\sum_N(2\pi)^4\delta^4(p_N-p)\langle\bar{0}|j^\alpha(0)|N\rangle\langle\bar{N}|\Phi(0)|0\rangle = 2\pi i\theta(+p_0)p^\alpha\rho(p^2)\;, \tag{4.17a}$$

$$\sum_N(2\pi)^4\delta^4(p_N-p)\langle\bar{0}|\Phi(0)|N\rangle\langle\bar{N}|j^\alpha(0)|0\rangle = 2\pi i\theta(+p_0)p^\alpha\bar{\rho}(p^2)\;. \tag{4.17b}$$

Moreover, causality requires that the commutator vanishes for space-like separations. If we choose $x^0 = y^0$, and $|\vec{x}-\vec{y}| > 0$, then it follows that $\rho(p^2) = -\bar{\rho}(p^2)$. Because the function is Lorenz invariant, this holds for all space-time configurations. We then arrive at the (Källén-Lehmann) spectral representation

$$\begin{aligned}\langle\bar{0}|[j^\alpha(y),\Phi(x)]|0\rangle &= \int\frac{d^4p}{(2\pi)^4}\left(2\pi i p^\alpha\theta\left(p_0\right)\rho\left(p^2\right)\left[e^{ip(x-y)}+e^{-ip(x-y)}\right]\right)\\ &= -\frac{\partial}{\partial y_\alpha}\int\frac{d^4p}{(2\pi)^4}\left(2\pi\theta\left(p_0\right)\rho\left(p^2\right)\left[e^{ip(x-y)}-e^{-ip(x-y)}\right]\right)\\ &= -\frac{\partial}{\partial y_\alpha}\int d\sigma^2\rho\left(\sigma^2\right)\Delta(y,x;\sigma^2)\;,\end{aligned} \tag{4.18}$$

where

$$\Delta(y,x;\sigma^2) \;=\; \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, 2\pi\,\theta(p_0)\,\delta(p^2-\sigma^2)\left[e^{ip\cdot(x-y)} - e^{-ip\cdot(x-y)}\right] \,, \qquad (4.19)$$

is the Pauli-Jordan function with the mass of the field replaced by σ. We know that this function should be zero for spacelike separations and non-zero for timelike separations.

Since the current is conserved, it follows that

$$-\Box_y \int \mathrm{d}\sigma^2\,\rho(\sigma^2)\Delta(y,x;\sigma^2) \;=\; \int \mathrm{d}\sigma^2\,\sigma^2\rho(\sigma^2)\Delta(y,x;\sigma^2) \;=\; 0\,. \qquad (4.20)$$

Since we know that for time-like seperations $\Delta(x-y) \neq 0$, the integrand must be zero as well

$$\sigma^2\rho(\sigma^2) = 0\,. \qquad (4.21)$$

From this it is straightforward to see that $\rho(\sigma^2) = \rho_0\delta(\sigma^2)$. Thus, for $x_0 = y_0$, we have

$$\begin{aligned}
\langle\bar{0}|[j^0(y),\Phi(x)]|0\rangle &= 2i\int d\sigma^2\rho\left(\sigma^2\right)\int \frac{d^4p}{(2\pi)^4}\theta\left(p^0\right)\delta\left(p^2-\sigma^2\right)p^0 e^{i\vec{p}(\vec{x}-\vec{y})} \\
&= i\int d\sigma^2\rho\left(\sigma^2\right)\int d\vec{p}e^{i\vec{p}(\vec{x}-\vec{y})} \\
&= i\,\delta^3(\mathbf{y}-\mathbf{x})\int d\sigma^2\rho\left(\sigma^2\right) = i\rho_0\,\delta^3(\mathbf{y}-\mathbf{x})\,, \qquad (4.22)
\end{aligned}$$

and it follows that

$$\langle\bar{0}|[Q,\Phi(x)]|0\rangle \;=\; iT\,\langle\Phi\rangle \;=\; i\rho_0\,. \qquad (4.23)$$

If there exists a non-trivial vacuum $\langle\Phi\rangle$, which is not invariant under the transformation generated by T, then $\rho_0 \neq 0$. We remark that, for a non-Hermitian theory, $\langle\Phi\rangle' = T\,\langle\Phi\rangle$ is a vacuum state of the transformed Lagrangian, e.g., for the transformations in equation (4.11), $\langle\Phi\rangle'$ is the vacuum state with respect to the potential in equation (4.13). The latter fact does not, however, affect the derivation of the

Goldstone theorem. Returning to the expressions in equation (4.17), we have

$$\sum_N (2\pi)^4 \delta^4(p_N - p)\langle\bar{0}|j^\nu(0)|N\rangle\langle\bar{N}|\Phi(0)|0\rangle = 2\pi i\theta(+p_0)p^\nu\rho_0\delta(p^2) \,. \quad (4.24)$$

The right-hand side is non-vanishing when $p^2 = 0$, provided $p^\nu \neq 0^\nu$. It follows that there must exist a state $|N\rangle$ with $p_N = p$, such that $p_N^2 = 0$, i.e. there must exist a massless state. □

We emphasise that this proof of the existence of a massless Goldstone mode relies on the existence of a conserved current and not on invariance of the Lagrangian. Hence, the Goldstone theorem persists for the non-Hermitian theory, and we give further details for our specific model in what follows.

4.1.3 Goldstone Mode

We have shown that a Nambu-Goldstone mode should exist for systems with a conserved current, whose transformation φ does not leave the vacuum invariant

$$\varphi\left(\langle\Phi\rangle\right) \neq \langle\Phi\rangle \,. \quad (4.25)$$

For the system we defined in (4.2) the non-trivial vacuum is given by

$$\langle\Phi\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sqrt{\frac{2}{\kappa}\left(m_1^2 - \frac{\mu^4}{m_2^2}\right)} \begin{pmatrix} 1 \\ \mu^2/m_2^2 \end{pmatrix} e^{i\varepsilon} \,, \; \varepsilon \in \mathbb{R} \,, \quad (4.26)$$

and such a transformation with conserved current is given by

$$\varphi : \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} e^{-i\gamma}\phi_1 \\ e^{i\gamma}\phi_2 \end{pmatrix} \,. \quad (4.27)$$

The Lagrangian (4.2) can then be expressed in terms of fluctuations around the non-trivial vacuum (4.26). This new Lagrangian is given by (4.8)

$$\begin{aligned}\mathscr{L} =&\partial_\alpha\hat{\phi}_1^*\partial^\alpha\hat{\phi}_1+\partial_\alpha\hat{\phi}_2^*\partial^\alpha\hat{\phi}_2+\frac{2\mu^4}{m_2^2}v_1\hat{\phi}_1-2m_2^2v_2\hat{\phi}_2-\tilde{m}_1^2|\hat{\phi}_1|^2-\frac{\kappa}{4}v_1^2\left(\hat{\phi}_1^2+(\hat{\phi}_1^*)^2\right)\\&-m_2^2|\hat{\phi}_2|^2-\mu^2\left(\hat{\phi}_1^*\hat{\phi}_2-\hat{\phi}_2^*\hat{\phi}_1\right)-\frac{\kappa}{2}v_1\left(\hat{\phi}_1^*+\hat{\phi}_1\right)|\hat{\phi}_1|^2-\frac{\kappa}{4}|\hat{\phi}_1|^4\,.\end{aligned}\tag{4.28}$$

The equations of motion of this Lagrangian follow from

$$\frac{\delta\mathscr{L}}{\delta\phi_i^*}\equiv 0\,,\ \text{ and, }\quad\left(\frac{\delta\mathscr{L}}{\delta\phi_i^*}\right)^*\equiv 0\,,\tag{4.29}$$

and are thus given by

$$\Box\begin{pmatrix}\hat{\phi}_1\\\hat{\phi}_1^*\\\hat{\phi}_2\\\hat{\phi}_2^*\end{pmatrix}=\begin{pmatrix}\frac{(m_1^2m_2^2-2\mu^4)}{m_2^2}&\frac{(m_1^2m_2^2-\mu^4)}{m_2^2}&\mu^2&0\\\frac{(m_1^2m_2^2-\mu^4)}{m_2^2}&\frac{(m_1^2m_2^2-2\mu^4)}{m_2^2}&0&\mu^2\\-\mu^2&0&m_2^2&0\\0&-\mu^2&0&m_2^2\end{pmatrix}\begin{pmatrix}\hat{\phi}_1\\\hat{\phi}_1^*\\\hat{\phi}_2\\\hat{\phi}_2^*\end{pmatrix}+\cdots\,,\tag{4.30}$$

where the dots represent terms of higher order in $\hat{\phi}_1$ and $\hat{\phi}_1^*$. These higher order terms can for now be ignored since they do not contribute to the masspectrum on the classical level.

One can straightforwardly check that the matrix for the linear terms in equation (4.30) has a determinant that is zero, and therefore has the anticipated Goldstone mode. Remark that even though the specific form of this matrix and of the classical eigenmodes does depend on our choice of equations of motion, the spectrum should be unique.

The mass matrix has a single zero eigenvalue, and the corresponding (Goldstone) mode is given by

$$G_1\ =\ \sqrt{\frac{2m_2^4}{m_2^4+\mu^4}}\left(\mathrm{Im}\,\hat{\phi}_1\ -\ \frac{\mu^2}{m_2^2}\,\mathrm{Im}\,\hat{\phi}_2\right)\,.\tag{4.31}$$

We also list the other eigenvalues of this system and their corresponding eigenmodes. The eigenvalues are given by

$$\lambda_2 = m_2^2 - \frac{\mu^4}{m_2^2}\,, \tag{4.32a}$$

$$\lambda_3 = \frac{1}{2m_2^2}\left(K + \sqrt{\left(K - 2m_2^4\right)^2 - 4\mu^4 m_2^4}\right)\,, \tag{4.32b}$$

$$\lambda_4 = \frac{1}{2m_2^2}\left(K - \sqrt{\left(K - 2m_2^4\right)^2 - 4\mu^4 m_2^4}\right)\,, \tag{4.32c}$$

with $K = 2m_1^2 m_2^2 - 3\mu^4 + m_2^4$. The corresponding eigenmodes are respectivly given by

$$G_2 = \sqrt{\left|\frac{2m_2^4}{m_2^4 - \mu^4}\right|}\left(\mathrm{Im}\,\hat{\phi}_2 - \frac{\mu^2}{m_2^2}\,\mathrm{Im}\,\hat{\phi}_1\right)\,, \tag{4.33a}$$

$$G_3 = \sqrt{2}\left|1 - \left(\frac{\mu^2}{\lambda_3 - m_2^2}\right)^2\right|^{-1/2}\left[\mathrm{Re}\,\hat{\phi}_1 + \left(\frac{\mu^2}{\lambda_3 - m_2^2}\right)\mathrm{Re}\,\hat{\phi}_2\right]\,, \tag{4.33b}$$

$$G_4 = \sqrt{2}\left|\left(\frac{\lambda_4 - m_2^2}{\mu^2}\right)^2 - 1\right|^{-1/2}\left[\mathrm{Re}\,\hat{\phi}_2 + \left(\frac{\lambda_4 - m_2^2}{\mu^2}\right)\mathrm{Re}\,\hat{\phi}_1\right]\,. \tag{4.33c}$$

A remarkable feature of the Goldstone theorem is that its form could also have been anticipated from the conserved current directly. This might initially seem surprising, but it reenforces the idea that it is the existence of this current that is responsible for the existence of the Goldstone mode.

The conservation equation yields

$$\partial_\alpha j^\alpha = i\partial_\alpha\left[\left(\phi_1^\star \partial^\alpha \phi_1 - \phi_1 \partial^\alpha \phi_1^\star\right) - \left(\phi_2^\star \partial^\alpha \phi_2 - \phi_2 \partial^\alpha \phi_2^\star\right)\right] = 0\,. \tag{4.34}$$

Expanding this to first order in the fluctuations gives

$$\partial_\nu j^\nu \simeq -2\left(v_1 \Box\,\mathrm{Im}\,\hat{\phi}_1 - v_2 \Box\,\mathrm{Im}\,\hat{\phi}_2\right)\,, \tag{4.35}$$

and we see that the Goldstone mode is

$$G_1 \propto \mathrm{Im}\,\hat{\phi}_1 - \frac{\mu^2}{m_2^2}\,\mathrm{Im}\,\hat{\phi}_2\,. \tag{4.36}$$

Remark that if we would have a conserved current whose transformation did not break the vacuum, these linear terms would be zero. So it follows that no Goldstone mode would exist.
Finally, we note that for our choice of equations of motion, the Goldstone mode is in fact the left eigenvector of the mass matrix (as dictated by the conserved current). Choosing the alternative definition of the variational procedure, the Goldstone mode would instead correspond to the right eigenvector of the mass matrix in equation (4.30), which is distinct and related to the previous one by $\mathscr{P}\mathscr{T}$-conjugation. Note that this is consistent with $\mathscr{P}\mathscr{T}$-transformation, superseding Hermitian conjugation for non-Hermitian theories and that the alternative definitions are equivalent.

4.2 Englert-Brout-Higgs Mechanism

We have shown that the Goldstone theorem still holds for a non-Hermitian system. The next step we want to take is to investigate how we can introduce gauge fields into the Lagrangian (4.2) in a consistent way. Since the Goldstone theorem still works for this Lagrangian, we want to eventually examine whether the Englert-Brout-Higgs mechanism also still applies to this non-Hermitian system.

The previous sections gave us a consistent way to generalise aspects from Hermitian Quantum Field Theory into non-Hermitian theories in a relatively straightforward manner. We will find that it is less straightforward to find a method that consistently gauges our non-Hermitian Lagrangian (4.2). We start this section by generalising a global transformation of the form

$$\phi \to e^{-i\gamma}\phi \ , \quad \text{where } \gamma \in \mathbb{R} \ , \tag{4.37}$$

to a local one. This will naturally introduce the gauge fields in our system and highlight the need for covariant derivatives in the gauged system. Afterwards, we should have the building blocks ready to start building a non-Hermitian gauged model. We will see that while doing so, we will need to take special care in making sure we have consistent equations of motion. This will closely follow the approach outlined in [76].

4.2.1 Local Symmetries

Gauge fields appear naturally when we want to make physical systems, possessing a gloabal symmetry, also invariant under a local symmetry. The main transformations we want to make local here are the global $U(1)$ transformations of the form (4.37) so that this becomes

$$\phi \to e^{-ief(x)}\phi \ , \tag{4.38}$$

where $f(x)$ can be any function that depends on space-time and e a charge constant. This will be a straightforward procedure for terms such as $m^2|\phi|^2$ that are locally well defined. When such terms are invariant under a global transformation of the form (4.37), it is straightforward to check that they will also be invariant under transformations such as (4.38). This is however less trivial when we deal with non-local properties such as kinetic terms. The derivative of a field $\phi(x)$ in the direction k^α is defined as

$$k^\alpha \partial_\alpha \phi \equiv \lim_{\delta \to 0} \frac{\phi(x+\delta k) - \phi(x)}{\delta} \ . \tag{4.39}$$

The problem with this property for systems with a local symmetry, is that taking the difference between fields that are evaluated at different points is badly defined. This is because $\phi(x)$ and $\phi(x+\delta k)$ can have completely different transformations under (4.38). We will need to find a way to consistently substract these fields that are evaluated at different points

$$\phi(x) - \phi(y) \quad \text{, with } x \neq y \ . \tag{4.40}$$

In order to do this, we introduce an opperator $U(x,y)$ that connects these different points in some way so that we can instead evaluate

$$\phi(x) - U(x,y)\phi(y) \ , \tag{4.41}$$

where $U(x,y)$ transforms under (4.38) as

$$U(x,y) \to e^{-ief(x)}U(x,y)e^{ief(y)} \ . \tag{4.42}$$

It is clear that for such a transformation it should be the case that $U(x,x) = 1$. Based on this, it is evident that instead of normal derivates as defined in (4.39) we should instead consider covariant derivatives defined as

$$k^\alpha D_\alpha \phi \equiv \lim_{\delta \to 0} \frac{\phi(x+\delta k) - U(x+\delta k,x)\phi(x)}{\delta} \ , \tag{4.43}$$

where we can express $U(x+\delta k,x)$ explicitly for infinitly small δ as

$$U(x+\delta k,x) = 1 - ie\delta k^\alpha A_\alpha(x) + \mathscr{O}\left(\delta^2\right) = \exp\left[-ie\int_x^{x+\delta k} A_\alpha(x)\, dx^\alpha\right] \ . \tag{4.44}$$

This prodecure naturally introduces a vector field A^α in the definition of the covariant derivative. It is easy to check that this vecor field should transform under a local transformation, such as (4.38) as

$$A^\alpha \to A^\alpha + \partial^\alpha f(x) \ . \tag{4.45}$$

This means that the covariant derivative must be defined as

$$D^\alpha \phi = [\partial^\alpha + ieA^\alpha]\,\phi \ . \tag{4.46}$$

The final step in constructing the Lagrangian will be the introduction of a kinetic term in our Lagrangian for the Gauge field of the form

$$-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = -\frac{1}{4}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) \ . \tag{4.47}$$

The definition of $U(x,y)$ is now only defined for x and y infinitly close. We can think about how this should be defined for x and y further apart. We know that $U(x,y)$ should be a function of the gauge field A^α and transform as (4.42). We can check that the expression

$$U_\gamma(x,y) = \exp\left(-ie\int_\gamma A_\alpha(x)\,dx^\alpha\right) , \tag{4.48}$$

with γ a path that runs form y to x, fits all the necessary criteria. This expression in known as a Wilson line [77] and depends on the path that connects y and x.

4.2.2 Gauging the Scalar Model

We are now able to make the Lagrangian

$$\mathscr{L} = \partial_\alpha\phi_1^*\partial^\alpha\phi_1 + \partial_\alpha\phi_2^*\partial^\alpha\phi_2 + m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2\left(\phi_1^*\phi_2 - \phi_2^*\phi_1\right) - \frac{g}{4}|\phi_1|^4 , \tag{4.49}$$

local $U(1)$ gauge invariant. There are two natuaral generalisations that we will follow using minimal substitution. The Lagrangians we will consider are given by

$$\begin{aligned}\mathscr{L}_\pm = \left[D_+^\alpha\phi_1\right]^* D_\alpha^+\phi_1 + \left[D_\pm^\alpha\phi_2\right]^* D_\alpha^\pm\phi_2 + m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2\left(\phi_1^*\phi_2 - \phi_2^*\phi_1\right) \\ - \frac{\kappa}{4}|\phi_1|^4 - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} ,\end{aligned} \tag{4.50}$$

where

$$D_\pm^\alpha = \partial^\alpha \pm iqA^\alpha . \tag{4.51}$$

Because of this definition of the covariant derivates, one can see that depending on the model, the gauge fields couple to different currents given by

$$j_{A,\pm}^\alpha = iq\left(\phi_1^* D_+^\alpha\phi_1 - \phi_1 D_+^\alpha\phi_1^*\right) \pm iq\left(\phi_2^* D_+^\alpha\phi_2 - \phi_2 D_+^\alpha\phi_2^*\right) . \tag{4.52}$$

In what follows we will discuss both of these models.

4.2.2.1 First Model: Conserved Current

If we want the Maxwell equations to have the usual canonical forms, so that the equations of motion for the gauge fields are given by

$$\partial_\alpha F^{\alpha\beta} = j_{A,-}^\beta . \tag{4.53}$$

From this equation of motion, it would follow that the current that couples to the gauge field must be conserved since $\partial_\alpha \partial_\beta F^{\alpha\beta} = 0$. The Lagrangian we consider should be of the form

$$\begin{aligned}\mathscr{L}_- =& [D^+_\alpha \phi_1]^\star D^\alpha_+ \phi_1 + [D^-_\alpha \phi_2]^\star D^\alpha_- \phi_2 + m_1^2 |\phi_1|^2 - m_2^2 |\phi_2|^2 - \mu^2 \left(\phi_1^\star \phi_2 - \phi_2^\star \phi_1\right) \\ & - \frac{\kappa}{4} |\phi_1|^4 - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \ . \end{aligned} \tag{4.54}$$

The kinetic terms in the Lagrangian are invariant under the transformations

$$\phi_1(x) \rightarrow \phi_1(x) e^{-iqf(x)} \ , \tag{4.55a}$$

$$\phi_2(x) \rightarrow \phi_2(x) e^{+iqf(x)} \ , \tag{4.55b}$$

$$A^\alpha(x) \rightarrow A^\alpha(x) + \partial^\alpha f(x) \ . \tag{4.55c}$$

The kinetic term could also be written in terms of $\mathscr{D}_\alpha \Phi$ with $\mathscr{D}_\alpha = \mathbb{I}_2 \partial_\alpha + iqPA_\alpha$, making manifest the role played by the parity matrix P in the definition of the conserved current.

For such a Lagrangian, we see that the non-Hermitian mass term explicitly breaks gauge invariance. Specifically, the gauge transformation yields a local mass squared matrix

$$M^2(x) = \begin{pmatrix} m_1^2 & \mu^2 e^{+2iqf(x)} \\ -\mu^2 e^{-2iqf(x)} & m_2^2 \end{pmatrix} \equiv \begin{pmatrix} m_1^2 & \tilde{\mu}^2(x) \\ [-\tilde{\mu}^2(x)]^\star & m_2^2 \end{pmatrix} \ . \tag{4.56}$$

The eigenspectrum is unaffected by the additional phases in the off-diagonal elements of equation (4.56), and the squared mass eigenvalues remain real and independent of the gauge function $f(x)$, since they involve $\tilde{\mu}^2(x)[\tilde{\mu}^2(x)]^\star = \mu^4$. While the eigenspectrum is gauge invariant, we find that the photon acquires a mass beyond tree-level; namely, at the one-loop level, we find that the polarisation tensor is not transverse:

$$k_\alpha \Pi^{\alpha\beta}(k^2 = 0) = \frac{q^2}{8\pi^2} \frac{k^\beta \mu^4}{(M_+^2 - M_-^2)^3} \left[M_+^4 - M_-^4 + 2M_+^2 M_-^2 \ln\left(\frac{M_-^2}{M_+^2}\right)\right] \ . \tag{4.57}$$

The above observations indicate that the non-Hermitian deformation of massless gauge theories is problematic due to the necessary violation of gauge invariance.

One can try to modify the Lagrangian (4.54) in order to still have a consistent theory where the gauge field still couples to a conserved current. One might be tempted to introduce a non-minimal coupling, with the Lagrangian

$$\begin{aligned}\mathscr{L}_W &= [D^+_\alpha \phi_1]^\star D^\alpha_+ \phi_1 + [D^-_\alpha \phi_2]^\star D^\alpha_- \phi_2 - m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 \\ &\quad -\mu^2\Big(W^2(x)\phi_1^\star\phi_2 - W^{*2}(x)\phi_2^\star\phi_1\Big) - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\ , \end{aligned} \tag{4.58}$$

where

$$W(x) \ = \ \exp\left[-iq\int^x A_\alpha \mathrm{d}y^\alpha\right] \tag{4.59}$$

is a Wilson line [77], running along a path from the boundary (at infinity) to the spacetime point x. Under a gauge transformation (chosen to vanish at infinity), we have

$$W(x) \ = W(x)e^{-iqf(x)}\ , \tag{4.60}$$

and the Lagrangian is invariant. However, we have traded the problem of gauge invariance for the path-dependence of the Wilson line. Moreover, we see that the gauge field now couples to the non-Hermitian term, such that the equation of motion for the gauge field obtains an imaginary part, potentially violating the reality of the gauge field.

4.2.2.2 Second Model: Modification of Charge Allocation

In order to circumvent the problems we discussed, we can instead couple the gauge field to the non-conserved current

$$j^\alpha_{A,+} = \ iq\big(\phi_1^\star D^\alpha_+\phi_1 - \phi_1[D^\alpha_+\phi_1]^\star\big) + iq\big(\phi_2^\star D^\alpha_+\phi_2 - \phi_2[D^\alpha_+\phi_2]^\star\big)\ , \tag{4.61}$$

where $D^\alpha_+ = \partial^\alpha + iqA^\alpha$, with divergence

$$\partial_\alpha j^\alpha_{A,+} = 2iq\mu^2(\phi_2^\star\phi_1 - \phi_1^\star\phi_2)\ . \tag{4.62}$$

The Lagrangian for this model is given by

$$\begin{aligned}\mathscr{L}_+ &= \left[D_+^\alpha \phi_1\right]^* D_\alpha^+ \phi_1 + \left[D_+^\alpha \phi_2\right]^* D_\alpha^+ \phi_2 + m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2\left(\phi_1^*\phi_2 - \phi_2^*\phi_1\right) \\ &\quad - \frac{\kappa}{4}|\phi_1|^4 - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \ . \end{aligned} \tag{4.63}$$

In this case, ϕ_1 and ϕ_2 are assigned identical charges and the non-Hermitian mass term is gauge invariant. This Lagrangian is invariant under the gauge transformation

$$\phi_1(x) \quad \to \quad \phi_1(x)e^{-iqf(x)} \ , \tag{4.64a}$$

$$\phi_2(x) \quad \to \quad \phi_2(x)e^{-iqf(x)} \ , \tag{4.64b}$$

$$A^\alpha(x) \quad \to \quad A^\alpha(x) + \partial^\alpha f(x) \ . \tag{4.64c}$$

However, in order to ensure that the Maxwell equations

$$\partial_\alpha F^{\alpha\beta} = j_{A,+}^\beta \ , \tag{4.65}$$

are consistent, since $\partial_\beta j_{A,+}^\beta \neq 0$, we need to add the term

$$-\frac{1}{2\xi}(\partial_\alpha A^\alpha)^2 \ , \tag{4.66}$$

to the Lagrangian, which would, in the Hermitian case, correspond to fixing a covariant gauge that satisfies the Lorenz gauge condition $\partial_\alpha A^\alpha = 0$. Notice that with the addition of this term and, as in the Hermitian case, the gauge functions must satisfy the constraint $\Box f = 0$ for the Lagrangian to be invariant under the transformation, such that we only have a restricted gauge invariance.

The equation of motion for the gauge field becomes

$$\Box A^\alpha - (1 - 1/\xi)\partial^\alpha \partial_\beta A^\beta = j_{A,+}^\alpha \ , \tag{4.67}$$

and its divergence yields

$$\frac{1}{\xi}\Box\partial_\alpha A^\alpha \;=\; 2iq\mu^2(\phi_2^\star\phi_1-\phi_1^\star\phi_2)\;. \tag{4.68}$$

We see that the non-Hermiticity precludes the Lorenz gauge condition, and the consistency of the Maxwell equation instead leads to the constraint

$$\Box\pi_0 \;=\; 2iq\mu^2(\phi_1^\star\phi_2-\phi_2^\star\phi_1)\;, \tag{4.69}$$

where $\pi_0=-\,\partial_\alpha A^\alpha/\xi$ is the momentum conjugate to A_0.

As a last remark, we note that the above formulation arises naturally from the Stückelberg mechanism [78] (see, e.g. [79]), in the limit where the vector mass goes to zero. To see this, we introduce an extra real scalar field ρ, and consider the Lagrangian

$$\begin{aligned}\mathscr{L}_\rho \;=\;& [D_\alpha\phi_1]^\star D^\alpha\phi_1+[D_\alpha\phi_2]^\star D^\alpha\phi_2-m_1^2|\phi_1|^2-m_2^2|\phi_2|^2-\mu^2\left(\phi_1^\star\phi_2-\phi_2^\star\phi_1\right)\\ &-\frac{\kappa}{4}|\phi_1|^4-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}+\frac{1}{2}\left(m_0A_\alpha-\partial_\alpha\rho\right)\left(m_0A^\alpha-\partial^\alpha\rho\right)\\ &-\frac{1}{2\xi}\left(\partial_\alpha A^\alpha+\xi m_0\rho\right)^2\;.\end{aligned} \tag{4.70}$$

This Lagrangian is invariant under the gauge transformations

$$\phi_{1,2}(x) \;\to\; \phi_{1,2}(x)e^{-iqf(x)}\;, \tag{4.71a}$$

$$A^\alpha(x) \;\to\; A^\alpha(x)+\partial^\alpha f(x)\;, \tag{4.71b}$$

$$\rho(x) \;\to\; \rho(x)+m_0 f(x)\;, \tag{4.71c}$$

where the gauge function satisfies $(\Box+\xi m_0^2)f=0$. The equation of motion for A_α then yields Eq. (4.67) in the limit $m_0\to 0$, where the scalar ρ decouples from the system, and the constraint (4.68) necessarily arises.

4.2.3 Englert-Brout-Higgs Mechanism

In this section, we show that a gauge-invariant mass can be generated at tree-level by the Englert-Brout-Higgs mechanism. Given the considerations in the previous section, we consider the Lagrangian

$$\begin{aligned}\mathscr{L} &= [D^{+}_{\alpha}\phi_1]^{\star}D^{\alpha}_{+}\phi_1+[D^{+}_{\alpha}\phi_2]^{\star}D^{\alpha}_{+}\phi_2+m_1^2|\phi_1|^2-m_2^2|\phi_2|^2-\mu^2\left(\phi_1^{\star}\phi_2-\phi_2^{\star}\phi_1\right)\\ &\quad -\frac{\kappa}{4}|\phi_1|^4-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}-\frac{1}{2\xi}\left(\partial_\alpha A^\alpha\right)^2 ,\end{aligned} \tag{4.72}$$

where we emphasise that the would-be gauge-fixing term $-(\partial_\alpha A^\alpha)^2/(2\xi)$ is necessary for consistency of the model.

The vacuum expectation value for the scalar fields is the same as in the global model (4.6), and we can express the Lagrangian (4.72) in terms of the shifted fields:

$$\begin{aligned}\mathscr{L} &= \partial_\alpha\hat{\phi}_1^{\star}\partial^\alpha\hat{\phi}_1+\partial_\alpha\hat{\phi}_2^{\star}\partial^\alpha\hat{\phi}_2-U(\hat{\phi}_1,\hat{\phi}_2)-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}-\frac{1}{2\xi}\left(\partial_\alpha A^\alpha\right)^2\\ &\quad +q^2A_\alpha A^\alpha\left(|v_1+\hat{\phi}_1|^2+|v_2+\hat{\phi}_2|^2\right)-A_\alpha j^\alpha_+ ,\end{aligned} \tag{4.73}$$

where $U(\hat{\phi}_1,\hat{\phi}_2)$ is defined as

$$U(\hat{\phi}_1,\hat{\phi}_2)=-\frac{2\mu^4}{m_2^2}v_1\hat{\phi}_1+2m_2^2v_2\hat{\phi}_2+\tilde{m}_1^2|\hat{\phi}_1|^2+\frac{\kappa}{4}v_1^2\left(\hat{\phi}_1^2+\left(\hat{\phi}_1^*\right)^2\right) \tag{4.74}$$

$$+m_2^2|\hat{\phi}_2|^2+\mu^2\left(\hat{\phi}_1^*\hat{\phi}_2-\hat{\phi}_2^*\hat{\phi}_1\right)+\frac{\kappa}{2}v_1\left(\hat{\phi}_1^*+\hat{\phi}_1\right)|\hat{\phi}_1|^2+\frac{\kappa}{4}|\hat{\phi}_1|^4 . \tag{4.75}$$

and j^α_+ as

$$j^\alpha_+=iq\left(\phi_1^*\partial^\alpha\phi_1-\phi_1\partial^\alpha\phi_1^*\right)+iq\left(\phi_2^*\partial^\alpha\phi_2-\phi_2\partial^\alpha\phi_2^*\right) . \tag{4.76}$$

We then obtain the equations of motion

$$\big(-D^2-\tilde{m}_1^2\big)\hat{\phi}_1 = +\mu^2\hat{\phi}_2 - q^2 v_1 A^2 + iqv_1\partial_\alpha A^\alpha \qquad (4.77a)$$
$$+\frac{\kappa}{2}v_1^2\hat{\phi}_1^* + \frac{\kappa}{2}\big(v_1\hat{\phi}_1^2 + 2v_1|\hat{\phi}_1|^2 + |\hat{\phi}_1|^2\hat{\phi}_1\big) ,$$
$$\big(-D^2-m_2^2\big)\hat{\phi}_2 = -\mu^2\hat{\phi}_1 - q^2 v_2 A^2 + iqv_2\partial_\alpha A^\alpha , \qquad (4.77b)$$
$$\big(-\Box - M_A^2\big)A^\alpha + (1-1/\xi)\partial^\alpha\partial_\beta A^\beta = 2q^2\big(v_1^\star\hat{\phi}_1 + v_1\hat{\phi}_1^\star + v_2^\star\hat{\phi}_2 + v_2\hat{\phi}_2^\star\big)A^\alpha$$
$$+2q^2\big(|\hat{\phi}_1|^2 + |\hat{\phi}_2|^2\big)A^\alpha - j_+^\alpha , \qquad (4.77c)$$

where

$$M_A^2 = 2q^2\left(|v_1|^2 + |v_2|^2\right) , \qquad (4.78)$$

is the gauge-invariant squared-mass of the gauge boson. Therefore, although the non-Hermitian model has non-trivial features related to gauge invariance, the usual Englert-Brout-Higgs mechanism still holds.

4.3 Alternative Discription

4.3.1 Equations of Motion

In the work done by Mannheim [58], an alternative approach to the equations of motion was presented. This approach to the equations of motion was also followed in [80] where the Goldstone bosons in onother scalar Quantum Field Theory. The non-Hermitian system transforms into a Harmitian one using a similarity transformation. In order to write this transformation explicitly, we first write the Lagrangian (4.2) in terms of the components

$$\phi_1 = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2) \quad , \quad \phi_1^* = \frac{1}{\sqrt{2}}(\chi_1 - i\chi_2) , \qquad (4.79a)$$
$$\phi_2 = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \quad , \quad \phi_2^* = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) . \qquad (4.79b)$$

Using this basis the action is writen as

$$S = \frac{1}{2}\int d^4x\Big[\partial_\alpha\chi_1\partial^\alpha\chi_1 + \partial_\alpha\chi_2\partial^\alpha\chi_2 + \partial_\alpha\psi_1\partial^\alpha\psi_1 + \partial_\alpha\psi_2\partial^\alpha\psi_2 + m_1^2\left(\chi_1^2+\chi_2^2\right)$$
$$-m_2^2\left(\psi_1^2+\psi_2^2\right) - 2i\mu^2\left(\chi_1\psi_2-\chi_2\psi_1\right) - \frac{\kappa}{8}\left(\chi_1^2+\chi_2^2\right)^2\Big] \; . \tag{4.80}$$

The canonical conjugates for the field ϕ_1 and ϕ_2 are repectively given as

$$\Pi_1\left(t,\vec{x}\right) = \partial_t\psi_1\left(t,\vec{x}\right) \quad , \quad \Pi_2\left(t,\vec{x}\right) = \partial_t\psi_2\left(t,\vec{x}\right) \; . \tag{4.81}$$

We can then define the transformations

$$T(\psi_1) = \exp\left[\frac{\pi}{2}\int d^3x\,\Pi_1(t,\vec{x})\psi_1(t,\vec{x})\right] \; , \tag{4.82a}$$

$$T(\psi_2) = \exp\left[\frac{\pi}{2}\int d^3x\,\Pi_2(t,\vec{x})\psi_2(t,\vec{x})\right] \; . \tag{4.82b}$$

Keeping in mind the equal time commutation relations

$$\left[\psi_i(t,\vec{x})\;,\;\Pi_1(t,\vec{x})\right] = i\delta^3\left(\vec{x}-\vec{y}\right) \; , \tag{4.83}$$

it follows that

$$T(\psi_1)\;\psi_1\;T^{-1}(\psi_1) = -i\psi_1 \; , \; T(\psi_1)\;\Pi_1\;T^{-1}(\psi_1) = i\Pi_1 \tag{4.84}$$

$$T(\psi_2)\;\psi_2\;T^{-1}(\psi_2) = -i\psi_2 \; , \; T(\psi_2)\;\Pi_2\;T^{-1}(\psi_2) = i\Pi_2 \; . \tag{4.85}$$

We can then apply these transformations to the action to obtain

$$T(\psi_1)T(\psi_2)\;S\;T^{-1}(\psi_2)T^{-1}(\psi_1) = S' \; , \tag{4.86}$$

where

$$S' = \frac{1}{2}\int d^4x\Big[\partial_\alpha\chi_1\partial^\alpha\chi_1 + \partial_\alpha\chi_2\partial^\alpha\chi_2 - \partial_\alpha\psi_1\partial^\alpha\psi_1 - \partial_\alpha\psi_2\partial^\alpha\psi_2 + m_1^2\left(\chi_1^2+\chi_2^2\right)$$
$$+m_2^2\left(\psi_1^2+\psi_2^2\right) - 2\mu^2\left(\chi_1\psi_2-\chi_2\psi_1\right) - \frac{\kappa}{8}\left(\chi_1^2+\chi_2^2\right)^2\Big] \; , \tag{4.87}$$

which is a Hermitian action. The equations of motion for this Hermitian Lagrangian are given by

$$\begin{cases} 0 &= \Box\phi_1 - m_1^2\phi_1 - i\mu^2\phi_2 + \frac{g}{2}|\phi_1|^2\phi_1 \ , \\ 0 &= \Box\phi_2 + m_2^2\phi_2 - i\mu^2\phi_1 \ , \end{cases} \tag{4.88a}$$

$$\begin{cases} 0 &= \Box\phi_1^* - m_1^2\phi_1^* + i\mu^2\phi_2^* + \frac{g}{2}|\phi_1|^2\phi_1^* \\ 0 &= \Box\phi_2^* + m_2^2\phi_2^* + i\mu^2\phi_1^* \ . \end{cases} \tag{4.88b}$$

Since these equations of motion stem from a Hermitian system, it is clear that they are consistent. Remark that the two systems S and S' are connected by a similarity transformation which leaves the eigenvalues invariant. Since S' is a Hermitian system, the Goldstone theorem will hold for this system. This proves the Goldstone theorem for our non-Hermitian system.

This procedure highlights a remarkable feature. It seems that, according to this procedure, some non-Hermitian systems are equivalent to the Hermitian one. An important question remains in how to interpret the seemingly incompatible equations of motion for the non-Hermitian system S, that are given by

$$\begin{cases} 0 &= \Box\phi_1 - m_1^2\phi_1 + \mu^2\phi_2 + \frac{g}{2}|\phi_1|^2\phi_1 \ , \\ 0 &= \Box\phi_2 + m_2^2\phi_2 - \mu^2\phi_1 \ , \end{cases} \tag{4.89a}$$

$$\begin{cases} 0 &= \Box\phi_1^* - m_1^2\phi_1^* - \mu^2\phi_2^* + \frac{g}{2}|\phi_1|^2\phi_1^* \\ 0 &= \Box\phi_2^* + m_2^2\phi_2^* + \mu^2\phi_1^* \ . \end{cases} \tag{4.89b}$$

A possible solution to this problem is to reinterpret the meaning of the star operator as the $\mathscr{CPT}$ conjugation, i.e. $\phi_i^* = \phi_i^{\mathscr{C}'\mathscr{PT}}$.

4.3.2 Gauge Symmetry in non-Hermitian System

The non-Hermitian Lagrangian can be made gauge invariant into the Lagrangian

$$\begin{aligned}\mathscr{L}_A = & [D_\alpha\phi_1]^* D^\alpha\phi_1 + [D_\alpha\phi_2]^* D^\alpha\phi_2 + m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 \\ & - \mu^2(\phi_1^*\phi_2 - \phi_2^*\phi_1) - \frac{\kappa}{4}|\phi_1|^4 - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} ,\end{aligned} \tag{4.90}$$

where $D^\alpha = \partial^\alpha + iqA^\alpha$. This Lagrangian is invariant under the gauge transformation

$$\phi_1 \to e^{-iqf(x)}\phi_1 \ , \ \phi_2 \to e^{-iqf(x)}\phi_2 \ , \ A^\alpha \to A^\alpha + \partial^\alpha f(x) . \tag{4.91}$$

Using the same basis as defined in (4.79), the non trivial vacuum expectation value is given by

$$\begin{aligned}\bar{\chi}_1 &= \frac{2}{\sqrt{\kappa}}\sqrt{m_1^2 - \mu^4/m_2^2} \ , \ \bar{\chi}_2 = 0 , \\ \bar{\psi}_2 &= \frac{-2i\mu^2}{\sqrt{\kappa}m_2^2}\sqrt{m_1^2 - \mu^4/m_2^2} \ , \ \bar{\psi}_1 = 0 .\end{aligned} \tag{4.92}$$

After expressing the Lagrangian $\mathscr{L}_A$ in terms of fluctuations around this vacuum, the gauge field A^α obtains the mass term

$$m_A^2 = \frac{4q^2}{\kappa}\frac{\left(m_1^2m_2^2 - \mu^4\right)\left(m_2^4 - \mu^4\right)}{m_2^6} . \tag{4.93}$$

Remark that we can do the same for the transformed action that is Hermitian and end up with the same mass term for the gauge field. Remark that this mass term is different from the one obtained from the original interpretation of the guage fields (4.78), which is given by

$$m_A^2 = \frac{4q^2}{\kappa}\frac{\left(m_1^2m_2^2 - \mu^4\right)\left(m_2^4 + \mu^4\right)}{m_2^6} . \tag{4.94}$$

These results predicts a very different behaviour in the limit $\mu^4 \to m_2^4$. In this limit the Englert-Brout-Higgs mechanism does no longer hold if we follow the approach as outlined by Mannheim [58]. The difference in these gauge masses is a direct result from how the equations of motion were obtained. Because of this, there is a difference in sign of μ^4 for the masses in the different approaches. This explains why, the limit $\mu^4 \to m_2^4$, our model [76] predict that the Englert-Brout-Higgs mechanism does still holds. We will examine this limit further in Chapter 6.

4.4 Conclusion

In conventional Hermitian Quantum Field Theories it is well understood how a global symmetry is accompanied by the appearance of a massless Nambu-Goldstone boson. In this chapter, we looked into how this connection is generalised when we also allow for non-Hermitian Quantum Field Theories. It turns out that the existence of the Nambu-Goldstone boson hinges on the presence of a transformation that breaks the non-trivial vacuum expectation value and has a corresponding conserved current. Such a transformation will in general not be a symmetry for non-Hermitian Quantum Field Theories. A consistent description of the Goldstone theorem for non-Hermitian Quantum Field Theories, serves as a first step into an exploration of whether consistent $\mathscr{PT}$-symmetric generalisation of the Standard Model can excist.

As a second step we, discussed how the Englert-Brout-Higgs mechanism [56, 57] for generating gauge boson masses can be generalised to the non-Hermitian case. We showed that in order to preserve gauge invariance in our non-Hermitian model, the gauge field should couple to a non-conserved current. To still have consistent Maxwell equations, we require the inclusion of gauge fixing terms in the Lagrangian. This leads to a particular constraint on the gauge field that depends on the non-Hermitian structure of the theory.

We ended this chapter by giving a short description of the work done by Mannheim in [58]. This work also looks into the Goldstone theorem and the Englert-Brout-Higgs mechanism for the same non-Hermitian model. The difference with our work lies in that Mannheim takes an alternative approach to defining the equations of motion. The interesting thing to note is that this procedure also predicts both the Goldstone theorem and the Englert-Brout-Higgs mechanism to hold, but the mass generated for the gauge field is different from the one we predicted before.

Chapter 5

Path Integral Quantisations

In the previous sections, all results have been derived at the tree-level. In order to define consistent Quantum Field Theories, we want to discuss these results at higher loop orders as well. We start this section by establishing a consistent formulation of the path integral. We will see that to do this, we need to define this with respect to the $\mathscr{C}'\mathscr{P}\mathscr{T}$-conjugate field variables. We are then able to define the partition function on the presence of two external $\mathscr{C}'\mathscr{P}\mathscr{T}$-conjugate source fields. Later, we move on to define the $1PI$ effective action for our scalar field Lagrangian and show the running of its couplings.
Later on, this procedure enables us to derive the Goldstone mode that we defined in the previous section at the one-loop order and to show that the gauge field A^α remains real after quantum corrections.

5.1 Path Integral Formulation

We now turn out attention to the formulation of the path integral representation of the non-Hermitian Quantum Field Theory. We discuss this for a scalar Lagrangian of the form

$$\mathscr{L} = \partial_\alpha \phi_1^* \partial^\alpha + \partial_\alpha \phi_2^* \partial^\alpha \phi_2 - m_1^2 |\phi_1|^2 - m_2^2 |\phi_2|^2 - \mu^2 \left(\phi_1^* \phi_2 - \phi_2^* \phi_1\right) - U_{\text{int}} \ , \quad (5.1)$$

for which the interaction potential U_{int} is $\mathscr{P}\mathscr{T}$-symmetric.

As we have seen, this Lagrangian has real eigenmasses squared as long as

$$\eta \equiv \frac{2\mu^2}{|m_1^2 - m_2^2|} \leq 1 \,, \tag{5.2}$$

with the eigenmasses squared given by

$$M_\pm = \frac{1}{2}\left(m_1^2 + m_2^2\right) \pm \frac{1}{2}\sqrt{\left(m_1^2 - m_2^2\right)^2 - 4\mu^4} \,. \tag{5.3}$$

For ease of notation and for the sake of consistency we will assume here that $m_1^2 \geq m_2^2$. Remark that we can always obtain this by relabeling the field and changing the sign of the μ^2 coupling. As we have discussed before, the physics of our system should remain invariant under this.

5.1.1 New Conjugate Field Variables

The Lagrangian in equation (5.1) would naively appear to have a finite imaginary part for $\mu \neq 0$, and one might be concerned that this could modify the convergence of the path integral. However, the spectrum of this theory is real and positive definite in the region of unbroken $\mathscr{P}\mathscr{T}$-symmetry, enabling us to consistently formulate the path integral and its quantisation.

Then, we can rotate to the mass eigenbasis via the transformation

$$\Xi \;\equiv\; R\Phi \;=\; \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \,, \qquad \bar{\Xi} \;\equiv\; \Phi^\dagger R^{-1} \;=\; \begin{pmatrix} \bar{\xi}_1 \\ \bar{\xi}_2 \end{pmatrix} \,, \tag{5.4}$$

where

$$R \;=\; \mathscr{N} \begin{pmatrix} \eta & 1-\sqrt{1-\eta^2} \\ 1-\sqrt{1-\eta^2} & \eta \end{pmatrix} \,, \tag{5.5}$$

with

$$\mathscr{N}^{-1} \;\equiv\; \sqrt{2\eta^2 - 2 + 2\sqrt{1-\eta^2}} \,. \tag{5.6}$$

The matrix R satisfies the following properties

$$R^\dagger \;=\; R \,, \qquad R^{-1} \;=\; PRP^{-1} \;=\; PRP \,, \tag{5.7}$$

with $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, such that

$$\bar{\Xi} \;=\; \Xi^{\ddagger} C' \,, \qquad \text{with} \qquad \Xi^{\ddagger} \;=\; \Xi^{\dagger} P \,, \qquad C' \;=\; RPR^{-1} \,. \tag{5.8}$$

The variables Ξ and $\bar{\Xi}$ are $\mathscr{C}'\mathscr{P}\mathscr{T}$-conjugate fields in the sense of what we discussed in Chapter 2. We note that the $\mathscr{C}'$ transformation here, which we identify with a prime, is thus not the canonical $\mathscr{C}$ transformation in Fock space, which would involve complex conjugation. Instead, it is the transformation by which one constructs the positive-definite inner product in $\mathscr{P}\mathscr{T}$-symmetric Quantum Mechanics as we discussed in Chapter 2. The free Lagrangian in terms of these fields becomes

$$\mathscr{L}_0 \;=\; \bar{\Xi}\Delta^{-1}\Xi \,, \qquad \text{where} \qquad \Delta^{-1} \;=\; \begin{pmatrix} -\Box - M_+^2 & 0 \\ 0 & -\Box - M_-^2 \end{pmatrix} \,, \tag{5.9}$$

and it appears to be that of an Hermitian theory. However, introducing interactions leads to the non-trivial feature mentioned above: varying the full action with respect to (ξ_1, ξ_2) or $(\bar{\xi}_1, \bar{\xi}_2)$ does not yield the same equations of motion. This can be seen, for example, with the interaction $|\phi_1\phi_1^\star|^2$, which can be expressed using either $\Phi = R^{-1}\Xi$:

$$|\phi_1\phi_1^\star|^2 \;=\; |\phi_1|^4 \;=\; \mathscr{N}^4 \big|\eta\,\xi_1 + \big(\sqrt{1-\eta^2} - 1\big)\xi_2\big|^4 \,, \tag{5.10}$$

or $\Phi^\dagger = \bar{\Xi}R$:

$$|\phi_1\phi_1^\star|^2 \;=\; |\phi_1^\star|^4 \;=\; \mathscr{N}^4 \big|\eta\,\bar{\xi}_1 - \big(\sqrt{1-\eta^2} - 1\big)\bar{\xi}_2\big|^4 \,. \tag{5.11}$$

5.1.2 Partition Function

The partition function is obtained from the vacuum persistence amplitude in the presence of external sources

$$J = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \qquad \text{and} \qquad \bar{J} = J^{\ddagger}C' \, . \tag{5.12}$$

For the non-Hermitian theory, this vacuum persistence amplitude is

$$Z[J,\bar{J}] = \langle \bar{0}(+\infty)|0(-\infty)\rangle_{J,\bar{J}} \, , \tag{5.13}$$

where the state $\langle \bar{0}|$ is the $\mathscr{C}'\mathscr{P}\mathscr{T}$ conjugate of the vacuum state. The path integral is developed in the usual way, except that one must insert complete sets of eigenstates of the Heisenberg-picture field operator Ξ and its $\mathscr{C}'\mathscr{P}\mathscr{T}$ conjugate $\bar{\Xi}$ (rather than its Hermitian conjugate) at all intermediate times. In this way, one arrives at the following result for the Euclidean path integral:

$$Z[J,\bar{J}] = \int \mathscr{D}[\Xi,\bar{\Xi}] \exp\left(- S_E[\Xi,\bar{\Xi}] + \int_x \left(\bar{J}\Xi + \bar{\Xi}J\right)\right) \, , \tag{5.14}$$

where S_E is the Euclidean action and we use the shorthand notation $\int_x \equiv \int \mathrm{d}^4x$. Of course, having established the correct form for the partition function, one could rewrite it in terms of the original $\mathscr{P}\mathscr{T}$-conjugate variables Φ and $\Phi^{\ddagger}$ by making the change of variables and accounting for the functional Jacobian, which is non-trivial but field independent.

The partition function (5.14) can be expanded around the free part

$$\begin{aligned} Z[J,\bar{J}] &= \int \mathscr{D}[\Xi,\bar{\Xi}] \; \exp\left[-\int_x \bar{\Xi}\Delta^{-1}\Xi + \int_x (\bar{J}\Xi + \bar{\Xi}J) - \int_x U_{\mathrm{int}} \right] \\ &= \exp\left[\int_x \bar{J}\Delta J \right] \int \mathscr{D}[\Pi,\bar{\Pi}] \; \exp\left[-\int_x \bar{\Pi}\Delta^{-1}\Pi - \int_x U_{\mathrm{int}} \right] \\ &= \exp\left[\int_x \bar{J}\Delta J \right] \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathscr{D}[\Pi,\bar{\Pi}] \; \exp\left[-\int_x \bar{\Pi}\Delta^{-1}\Pi \right] \left[-\int_x U_{\mathrm{int}} \right]^n , \end{aligned} \tag{5.15}$$

where $\Delta^{-1} = \mathrm{diag}\,(-\partial^2 + M_+^2, -\partial^2 + M_-^2)$ in Euclidean signature and $\Pi \equiv \Xi - \Delta J = (\pi_1\,,\pi_2)^{\mathsf{T}}$. One can see that the perturbative structure is the usual one, comprising well-defined Gaussian integrals at each order.

5.1.3 One-loop 1PI Effective Action

There is an unambiguous definition of the classical saddle point $(\Xi_0, \bar{\Xi}_0)$ for the path integral (5.14), which satisfies

$$\left[-\frac{\delta S_E}{\delta \Xi} + \bar{J}\right]_0 = 0 = \left[-\frac{\delta S_E}{\delta \bar{\Xi}} + J\right]_0 , \tag{5.16}$$

where the index 0 indicates evaluation at the configuration $(\Xi_0, \bar{\Xi}_0)$. Expanding the partition function up to quadratic order around the saddle point, we obtain for the one-loop partition function

$$\begin{aligned} Z^{(1)}[J,\bar{J}] &= \exp\left[-S_E[\Xi_0,\bar{\Xi}_0] + \int_x \Big(\bar{J}\Xi_0 + \bar{\Xi}_0 J\Big)\right] \\ &\quad \times \int \mathscr{D}[\Xi,\bar{\Xi}]\, \exp\left[-\frac{1}{2}\int_{xy}\Big(2\,(\bar{\Xi}-\bar{\Xi}_0)_x \frac{\delta^2 S_E}{\delta\bar{\Xi}_x \delta\Xi_y}\Big|_0 (\Xi-\Xi_0)_y \right. \\ &\quad \left. + (\bar{\Xi}-\bar{\Xi}_0)_x \frac{\delta^2 S_E}{\delta\bar{\Xi}_x\delta\bar{\Xi}_y}\Big|_0 (\bar{\Xi}-\bar{\Xi}_0)_y + (\Xi-\Xi_0)_x \frac{\delta^2 S_E}{\delta\Xi_x\delta\Xi_y}\Big|_0 (\Xi-\Xi_0)_y\Big)\right] \\ &= \exp\left[-S_E[\Xi_0,\bar{\Xi}_0] + \int_x\Big(\bar{J}\Xi_0 + \bar{\Xi}_0 J\Big) - \frac{1}{2}\,\mathrm{STr}\,\ln S_E^{(2)}\Big|_0\right], \end{aligned} \tag{5.17}$$

where $S_E^{(2)}$ is the functional Hessian matrix (in field space) of the Euclidean action and STr indicates the trace over both coordinate and field spaces. In order to define the one-particle irreducible (1*PI*) effective action $\Gamma^{(1)}$, one introduces the background field Ξ_c:

$$\Xi_c = \frac{1}{Z^{(1)}}\frac{\delta Z^{(1)}}{\delta \bar{J}} , \tag{5.18}$$

which, from equation (5.17), is

$$\begin{aligned}\Xi_c &= \Xi_0 + \int_x \left(-\frac{\delta S_E}{\delta \Xi_0} + \bar{J} \right) \frac{\delta \Xi_0}{\delta \bar{J}} - \frac{1}{2} \frac{\delta}{\delta \bar{J}} \mathrm{STr} \ln S_E^{(2)} \Big|_0 \\ &= \Xi_0 + \text{quantum corrections} \ . \end{aligned} \tag{5.19}$$

$\Gamma^{(1)}$ is then defined after inverting the relation (5.18) to express $\bar{J}$ as a functional of Ξ_c:

$$\begin{aligned}\Gamma^{(1)}[\Xi_c, \bar{\Xi}_c] &= -\ln Z^{(1)} + \int_x \left(\bar{J}\Xi_c + \bar{\Xi}_c J \right) \\ &= S_E[\Xi_c, \bar{\Xi}_c] + \frac{1}{2} \mathrm{STr} \ln S_E^{(2)} \Big|_c \ , \end{aligned} \tag{5.20}$$

were the index c indicates evaluation in the background field configuration. The one-loop $1PI$ effective potential is obtained for a constant configuration Ξ_c and is then given by

$$U^{(1)}(\Xi_c, \bar{\Xi}_c) = U(\Xi_c, \bar{\Xi}_c) + \frac{1}{2V^{(4)}} \mathrm{STr} \ln S_E^{(2)} \Big|_c \ , \tag{5.21}$$

where $V^{(4)}$ is the spacetime volume. After a rotation to the original basis, which does not affect the trace, we finally obtain

$$U^{(1)}(\Phi_c, \Phi_c^\dagger) = U(\Phi_c, \Phi_c^\dagger) + \frac{1}{2V^{(4)}} \mathrm{STr} \ln S_E^{(2)} \Big|_c \ . \tag{5.22}$$

5.1.4 Running Couplings

After deriving the one-loop $1PI$ effective action, we are able to discuss the running of the couplings for our scalar field. We consider here a bare interaction potential of the form

$$\begin{aligned}U_{\mathrm{int}}^{(0)} &= \frac{g_1}{4}|\phi_1|^4 + \frac{g_2}{4}|\phi_2|^4 + \lambda|\phi_1\phi_2|^2 + \frac{\alpha}{4}\left((\phi_1^\star\phi_2)^2 + (\phi_2^\star\phi_1)^2 \right) \\ &\quad + \frac{1}{2}\left(\beta_1|\phi_1|^2 + \beta_2|\phi_2|^2 \right)\left(\phi_1^\star\phi_2 - \phi_2^\star\phi_1 \right) \ . \end{aligned} \tag{5.23}$$

Substituting this potential into equation (5.22) leads to the following one-loop running of the coupling constants (details can be found in the Appendix A.1):

$$\left(m_i^2\right)^{(1)} = m_i^2 + \frac{g_i+\lambda}{16\pi^2}\Lambda^2 + \mathscr{O}\left(\ln\left(\frac{\Lambda}{m}\right)\right) , \tag{5.24a}$$

$$\left(\mu^2\right)^{(1)} = \mu^2 + \frac{\beta_1+\beta_2}{16\pi^2}\Lambda^2 - \frac{1}{8\pi^2}\left(\mu^2(\lambda-\alpha)+\beta_1 m_1^2+\beta_2 m_2^2\right)\ln\left(\frac{\Lambda}{m}\right) , \tag{5.24b}$$

$$g_i^{(1)} = g_i - \frac{1}{16\pi^2}\left(5g_i^2+\alpha^2+4\lambda^2-10\beta_i^2\right)\ln\left(\frac{\Lambda}{m}\right) , \tag{5.24c}$$

$$\lambda^{(1)} = \lambda - \frac{1}{16\pi^2}\left(4\lambda^2+2\alpha^2+2\lambda\left(g_1+g_2\right)-3\left(\beta_1^2+\beta_2^2\right)-4\beta_1\beta_2\right)\ln\left(\frac{\Lambda}{m}\right) , \tag{5.24d}$$

$$\alpha^{(1)} = \alpha - \frac{1}{16\pi^2}\left(4\left(\beta_1^2+\beta_2^2\right)+\alpha\left(g_1+g_2\right)+2\beta_1\beta_2+8\lambda\alpha\right)\ln\left(\frac{\Lambda}{m}\right) , \tag{5.24e}$$

$$\beta_i^{(1)} = \beta_i - \frac{1}{16\pi^2}\left(5g_i\beta_i+4\beta_j\lambda-\alpha\beta_j+6\lambda\beta_i-4\alpha\beta_i\right)\ln\left(\frac{\Lambda}{m}\right) , \tag{5.24f}$$

where m is a typical mass scale of the system, $i \neq j$, and finite terms are omitted.

5.1.5 Hermitian Fixed Point

We assume here that the non-Hermitian interactions are switched off ($\beta_i = 0$) and the only source of non-Hermiticity is the mass parameter μ^2. Quantum corrections modify this mass parameter, and we need to check that the condition (5.2), which delineates the phase of unbroken $\mathscr{PT}$-symmetry, remains valid at one loop. For a fixed set of dressed parameters, the one-loop running of the parameter η is

$$\eta(\Lambda) = \left|\frac{2(\mu^2)^{(1)}-(\alpha^{(1)}-\lambda^{(1)})\mu^2/(4\pi^2)\ln(\Lambda/m))}{(m_1^2)^{(1)}-(m_2^2)^{(1)}-\left(g_1^{(1)}-g_2^{(1)}\right)\Lambda^2/(16\pi^2)}\right| . \tag{5.25}$$

We recall that, for the $\mathscr{PT}$-symmetry to be unbroken, the following requirement needs to be satisfied for all values of Λ:

$$\eta(\Lambda) < 1 . \tag{5.26}$$

If $g_1^{(1)} \neq g_2^{(1)}$, we can see that $\eta(\Lambda) \to 0$ when $\Lambda \to \infty$, such that the theory converges to a Hermitian limit, which thus appears as a UV fixed point.

5.2 Beyond tree-level Calculations

In this subsection, we use the procedure we just outlined to calculate path integrals, to explicitly do some beyond tree-level calculations. Firstly, we calculate the Goldstone modes at the one-loop level. After this, we use the path integral description to make sure that the gauge field we introduced in Section 4.2.2 remains real after including quantum correction.

5.2.1 The Goldstone Mode to one-loop Order

Previously in Chapter 4, we have shown that the Goldstone theorem still holds for our non-Hermitian Quantum Field Theory. We derived the spectrum for the Lagrangian

$$\mathscr{L} = \partial_\alpha \phi_1^* \partial^\alpha \phi_1 + \partial_\alpha \phi_2^* \partial^\alpha \phi_2 + m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2\left(\phi_1^*\phi_2 - \phi_2^*\phi_1\right) - \frac{\kappa}{4}|\phi_1|^4\,, \tag{5.27}$$

after spontaneous symmetry breaking, at tree-level. We can now use the equation (5.22) to derive the eigenmodes at the one-loop order. The full tree-level potential is given in terms of the fields $\hat{\phi}_1$ and $\hat{\phi}_2$ as

$$\begin{aligned} U^{(0)} &= M_1^2|\hat{\phi}_1|^2 + m_2^2\hat{\phi}_2\left(\hat{\phi}_2^\star + M_c\right) + \mu^2\Big(\hat{\phi}_2\hat{\phi}_1^\star - \hat{\phi}_1\left(\hat{\phi}_2^\star + M_c\right)\Big) \\ &\quad + \frac{M_a^2}{2}\Big(\hat{\phi}_1^2 + \left(\hat{\phi}_1^\star\right)^2\Big) + \frac{M_b}{2}|\hat{\phi}_1|^2\left(\hat{\phi}_1^\star + \hat{\phi}_1\right) + \frac{\kappa}{4}|\hat{\phi}_1|^4\,, \end{aligned} \tag{5.28}$$

where we use the notation

$$
\begin{aligned}
M_1^2 &= \frac{m_1^2 m_2^2 - 2\mu^4}{m_2^2} \,, && (5.29a)\\
M_a^2 &= \frac{m_1^2 m_2^2 - \mu^4}{m_2^2} \;=\; M_1^2 + \frac{\mu^4}{m_2^2} \,, && (5.29b)\\
M_b &= g\sqrt{2\frac{m_1^2 m_2^2 - \mu^4}{\kappa m_2^2}} \,, && (5.29c)\\
M_c &= \frac{2\mu^2}{m_2^2}\sqrt{2\frac{m_1^2 m_2^2 - \mu^4}{\kappa m_2^2}} \,. && (5.29d)
\end{aligned}
$$

As discussed before, the linear terms in the potential are a consequence of the non-Hermitian nature of the system. At the one-loop level, these couplings are obtained by substituting this potential into equation (5.22) and are given by

$$
\begin{aligned}
\kappa^{(1)} &= \kappa - \frac{5\kappa^2}{16\pi^2}\ln\left(\frac{\Lambda}{m}\right) \,, && (5.30a)\\
m_2^{2(1)} &= m_2^2 \,, && (5.30b)\\
\mu^{2(1)} &= \mu^2 \,, && (5.30c)\\
M_1^{2(1)} &= M_1^2 + \frac{\kappa\Lambda^2}{16\pi^2} + \mathscr{O}\left(\ln\left(\Lambda/m\right)\right) \,, && (5.30d)\\
M_a^{2(1)} &= M_a^2 - \frac{1}{16\pi^2}\left(2M_b^2 + \kappa M_a^2\right)\ln\left(\frac{\Lambda}{m}\right) \,, && (5.30e)\\
M_b^{(1)} &= M_b - \frac{5\kappa M_b}{16\pi^2}\ln\left(\frac{\Lambda}{m}\right) \,, && (5.30f)
\end{aligned}
$$

where finite terms are again omitted. A linear term is also generated, which is given by

$$
M_b\frac{\Lambda^2}{16\pi^2}\left(\hat{\phi}_1^\star + \hat{\phi}_1\right) \,, \qquad (5.31)
$$

so that the one-loop potential in terms of $\hat{\phi}_1$, $\hat{\phi}_2$ becomes

$$\begin{aligned}U^{(1)} &= M_b \frac{\Lambda^2}{16\pi^2}\left(\hat{\phi}_1+\hat{\phi}_1^\star\right) + m_2^2 M_c \hat{\phi}_2 - \mu^2 M_c \hat{\phi}_1 + \left(M_1^2+\frac{\kappa\Lambda^2}{16\pi^2}\right)|\hat{\phi}_1|^2 + m_2^2|\hat{\phi}_2|^2 \\ &\quad + \mu^2\left(\hat{\phi}_2\hat{\phi}_1^\star-\hat{\phi}_1\hat{\phi}_2^\star\right) + \left(\frac{M_a^2}{2}-\left(M_b^2+\frac{\kappa M_a^2}{2}\right)\frac{\ln\left(\frac{\Lambda}{m}\right)}{16\pi^2}\right)\left(\hat{\phi}_1^2+\left(\hat{\phi}_1^\star\right)^2\right) \\ &\quad + \frac{M_b}{2}\left(1-\frac{5g\ln\left(\frac{\Lambda}{m}\right)}{16\pi^2}\right)|\hat{\phi}_1|^2\left(\hat{\phi}_1^\star+\hat{\phi}_1\right) + \frac{\kappa}{4}\left(1-\frac{5g\ln\left(\frac{\Lambda}{m}\right)}{16\pi^2}\right)|\hat{\phi}_1|^4 .\end{aligned} \tag{5.32}$$

To show the existence of the Goldstone mode to one-loop order, we should express the fields in terms of fluctuations around the new shifted vacuum. From this, we can find the one-loop-corrected vevs

$$\begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \left(1-\frac{\kappa}{2M_a^2}\frac{\Lambda^2}{16\pi^2}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} , \tag{5.33}$$

where v_1, v_2 were defined in equation (4.6). Expressing the one-loop potential in terms of the fields fluctuating around this minimum

$$\phi_1 = v_1^{(1)} + \hat{\phi}_1^{(1)} \qquad \text{and} \qquad \phi_2 = v_2^{(1)} + \hat{\phi}_2^{(1)} \tag{5.34}$$

gives equations of motion of the form

$$\Box \begin{pmatrix} \hat{\phi}_1^{(1)} \\ \left(\hat{\phi}_1^{(1)}\right)^\star \\ \hat{\phi}_2^{(1)} \\ \left(\hat{\phi}_2^{(1)}\right)^\star \end{pmatrix} = \begin{pmatrix} M_1^2-\frac{\kappa\Lambda^2}{16\pi^2} & M_a^2-\frac{\kappa\Lambda^2}{16\pi^2} & \mu^2 & 0 \\ M_a^2-\frac{\kappa\Lambda^2}{16\pi^2} & M_1^2-\frac{\kappa\Lambda^2}{16\pi^2} & 0 & \mu^2 \\ -\mu^2 & 0 & m_2^2 & 0 \\ 0 & -\mu^2 & 0 & m_2^2 \end{pmatrix}\begin{pmatrix} \hat{\phi}_1^{(1)} \\ \left(\hat{\phi}_1^{(1)}\right)^\star \\ \hat{\phi}_2^{(1)} \\ \left(\hat{\phi}_2^{(1)}\right)^\star \end{pmatrix} + \cdots . \tag{5.35}$$

The mass matrix again has determinant zero, showing that we still have a Goldstone mode at one-loop order, which is given by

$$G_1^{(1)} = \sqrt{\frac{2m_2^4}{m_2^4+\mu^4}}\left(\operatorname{Im}\hat{\phi}_1^{(1)} - \frac{\mu^2}{m_2^2}\operatorname{Im}\hat{\phi}_2^{(1)}\right) . \tag{5.36}$$

We see that the one-loop Goldstone mode is related to the Goldstone mode at tree level; the one-loop mode is obtained from the tree-level one simply by making the replacement $\hat{\phi}_1, \hat{\phi}_2 \to \hat{\phi}_1^{(1)}, \hat{\phi}_2^{(1)}$.

5.2.2 Reality of the Background Gauge Field

When we gauge our non-Hermitian scalar Lagrangian, we want to make sure that the gauge field remains real after taking quantum corrections into account. We do this by using the Euclidean partition function Z. This partition function must be $\mathscr{C}'\mathscr{P}\mathscr{T}$-symmetric, as discussed previously, and is defined as

$$Z = \int \mathscr{D}[A_\alpha, \Xi, \bar{\Xi}] \exp\left(-S_E\left[\Xi, \bar{\Xi}\right] + \int_x \left(J_\alpha A^\alpha + \bar{\chi} \cdot \Xi + \bar{\Xi} \cdot \chi\right)\right) , \tag{5.37}$$

where χ and $\bar{\chi}$ are the sources for the scalar fields Ξ and $\bar{\Xi}$ and J_α the source for the gauge field A^α. The background gauge field A_b^α is defined as

$$A_b^\alpha = \frac{1}{Z} \frac{\delta Z}{\delta J_\alpha} , \tag{5.38}$$

For A_b^α to be real, it is enough to find a condition for the Euclidean partition function to be real, although the Euclidean action S_E has an imaginary part, which is opposite in sign to $\mathrm{Im}\, S$, given by

$$\mathrm{Im} S = i\mu^2 \int d^4x \left(\phi_1^* \phi_2 - \psi_2^* \phi_1\right) . \tag{5.39}$$

This condition can be achieved by choosing the transformation of the sources χ_k under $\mathscr{P}\mathscr{T}$ appropriately. For this, we note that the partition function can also be written as

$$Z = \int \mathscr{D}[A_\alpha, \Xi, \bar{\Xi}] \exp\left(-S_E\left[\Xi, \bar{\Xi}\right] + \int_x \left(J_\alpha A^\alpha + \bar{\chi} \cdot R \cdot \Phi + \Phi^\dagger \cdot R^{-1} \cdot \chi\right)\right) , \tag{5.40}$$

such that

$$Z^{\star}=\int\mathscr{D}[A_{\alpha},\Xi,\bar{\Xi}]\exp\left(-S_{E}^{\star}\left[\Xi,\bar{\Xi}\right]+\int_{x}\left(J_{\alpha}A^{\alpha}+\Phi^{\dagger}\left[PR^{-1}P\right](\bar{\chi})^{\dagger}+\chi^{\dagger}\left[PR\,P\right]\Phi\right)\right), \tag{5.41}$$

which, after the change of variable $\Phi\to P\,\Phi$, leads to

$$Z^{\star}=\int\mathscr{D}[A_{\alpha},\Xi,\bar{\Xi}]\exp\left(-S_{E}^{\star}\left[\Xi,\bar{\Xi}\right]+\int_{x}\left(J_{\alpha}A^{\alpha}+\Phi^{\dagger}\left[R^{-1}P\right](\bar{\chi})^{\dagger}+\chi^{\dagger}\left[PR\right]\Phi\right)\right), \tag{5.42}$$

Imposing $Z^{\star}=Z$ implies then that $\bar{\chi}=(P\chi)^{\dagger}$. As a consequence, $\mathscr{PT}$ symmetry ensures that the gauge field remains real after quantum corrections, even though it is coupled to a non-Hermitian scalar sector.

Finally, one can also conclude from the reality of the partition function that physical observables depend on μ^4 only. Indeed, for Z to be real, the imaginary part of the action, cf. equation (5.39), must contribute to the calculation of Z with even powers, and thus with $(\pm\mu^2)^2$. This property, predicted at the tree-level can, therefore, be extended to the full quantum system.

5.3 Conclusion

In this section, we discussed the path integral description of our non-Hermitian $\mathscr{PT}$-symmetric system. To define a consistent path integral definition, we need to make sure we integrate over $\mathscr{C}'\mathscr{PT}$-conjugate variables instead of Hermitian conjugate variables. This makes physical sense since we should expect the partition function Z of a $\mathscr{C}'\mathscr{PT}$-symmetric system to also be $\mathscr{C}'\mathscr{PT}$-symmetric. This procedure enables us to consistently define the $1PI$ effective action of our theory and show how the couplings of our non-Hermitian, $\mathscr{PT}$-symmetric Lagrangian run for a general $\mathscr{PT}$-symmetric quartic interaction. This procedure shows us that the non-Hermitian mass coupling grows because of the non-Hermitian quartic interaction couplings in the UV limit. When we consider a case where the interactions are Hermitian, our theory has a Hermitian UV fixed point.

Next we are able to put this procedure to the test. We use the one-loop $1PI$ effective action to calculate the eigenmasses squared for a potential given by (5.28). This allows us again to confirm the existence of the Goldstone mode and calculate its explicitly at one-loop. Secondly, we check that after coupling our scalar model to a vector field after gauging our model, the vector field remains real even when we take quantum correction into account. It turns out that the reality of the gauge field and the partition function is assured if we derive the correct $\mathscr{C}'\mathscr{P}\mathscr{T}$ transformation properties for the scalar source fields. It turns out that the definition of the partition functions is consistent and we are able to confirm the existence of the Nambu-Goldstone mode at one-loop order and prove the reality of our vector field.

Chapter 6

Non-Abelian Spontaneous Symmetry Breaking

We previously discussed $U(1)$ gauge invariance for our non-Hermitian system. There we described how this local symmetry requires the existence of a $U(1)$ gauge field. Next, we want to study how to include $SU(2)$ gauge fields into our system and consistently impose $SU(2)$ gauge invariance. For the fields in Lagrangian (4.2) to transform under a $SU(2)$ transformations, we will upgrade these scalar fields ϕ_i into scalar doublets of the form Φ_i. Doing so, we naturally end up with a two-Higgs-doublet model.

As was the case for the Abelian gauge symmetry, we need to couple the gauge fields to non-conserved currents to obtain consistent equations of motion. To achieve consistent Maxwell equations, we will again have to impose gauge restrictions. The explicit form of these restrictions are found by using the BRST symmetry of our system. After having derived a consistent $U(1)\times SU(2)$ invariant theory, we discuss the Englert-Brout-Higgs mechanism for our non-Hermitian two-Higgs-doublet model. We then derive both the scalar mass spectrum and the gauge mass spectrum and compare these results to those obtained by a Hermitian two-Higgs-doublet model. We show that the masses we get, all depend on μ^4. This supports the consistency of the defenition the equations of motion as we discussed in section (3.1.2).

This Chapter has the following outline. Firstly, we discuss the Hermitian two-Higgs-doublet model with a Hermitian doublet mixing term. We derive the physical spectrum for this theory, and find that the system consists out of three massless Goldstone field, two charged Higgs bosons and three neutral Higgs bosons. After this, we consider a non-Hermitian two-Higgs-doublet model with an anti-Hermitian doublet mixing term. We discuss how to consistently gauge this model and derive the spectrum of this model. Similar to the Hermitian model, we find a charged and a neutral Goldstone field and two charged and three neutral Higgs bosons. Finally, we compare the spectrum of both of these models. From this we see that the spectrum differs significantly. It is however clear that this non-Hermitian model serves as an analytic continuation of the Hermitian model. In this chapter, we follow work as outlined in [81].

6.1 Hermitian Two-Higgs-doublet Model

The Higgs sector of the Standard Model Lagrangian before symetry breaking [82], [83] is given by

$$\mathscr{L} = [D^{\alpha}\Phi]^{\dagger} D_{\alpha}\Phi + m^2|\Phi|^2 - \lambda|\Phi|^4 \ , \tag{6.1}$$

where Φ is a complex doublet and D^{α} an appropriate covariant derivative. We can consider a straightforward extension to the Standard Model by including extra scalar fields. These models are known as multi-Higgs-doublet models (2HDM) and are studied in works such as [60, 84]. We are mostly interested in two-Higgs-doublet models [59, 61, 62].

We have discussed gauge invariance for an Abelian transformation of the form

$$\phi_i(x) \to e^{-if(x)}\phi_i(x) \ , \tag{6.2}$$

for our non-Hermitian system with two complex scalar fields. As a next step, we want to discuss non-Abelian $U(1) \times SU(2)$ gauge invariance in our non-Hermitian system. In doing so, we upgrade our scalar fields to doublets and naturally obtain a non-Hermitian two-Higgs-doublet model. Before we start discussing this model, we look into a Hermitian two-Higgs-doublet model and its features.

We discuss a two-Higgs-doublet model as described in works such as [59, 61, 62]. Remark that in this section, we are only interested in the scalar sector and thus do not consider the Yukawa coupling $\mathscr{L}_Y$ of the form

$$\mathscr{L}_Y = \sum_{i,j} \left(g_{ij}^{(u)} \overline{\Psi}_{Li} \Phi_1 u_{Rj} + g_{ij}^{(d)} \overline{\Psi}_{Li} \Phi_2 d_{Rj} \right) , \tag{6.3}$$

that couples the Higgs bosons and the quarks or the covariant derivatives that couples the scalar fields to the gauge fields. The most general scalar two-Higgs-doublet model has 14 parameters. In most models, these parameters are reduced by assuming certain symmetries. We study a potential of the form

$$\begin{aligned} V =& m_1^2 |\Phi_1|^2 + m_2^2 |\Phi_2|^2 - m_{12}^2 \left(\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1 \right) + \frac{\kappa_1}{4} |\Phi_1|^4 + \frac{\kappa_2}{4} |\Phi_2|^4 \\ &+ \kappa_3 |\Phi_1|^2 |\Phi_2|^2 + \kappa_4 |\Phi_1^\dagger \Phi_2|^2 + \frac{\kappa_5}{2} \left(\left[\Phi_1^\dagger \Phi_2 \right]^2 + \left[\Phi_2^\dagger \Phi_1 \right]^2 \right) , \end{aligned} \tag{6.4}$$

where Φ_i are complex doublets given by

$$\Phi_1 = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \quad , \quad \Phi_2 = \begin{pmatrix} \phi_5 + i\phi_6 \\ \phi_7 + i\phi_8 \end{pmatrix} . \tag{6.5}$$

This potential consists out of a $U(1) \times SU(2)$ invariant, $\mathscr{C}\mathscr{P}$ even and even under $\Phi_1 \to \Phi_1, \Phi_2 \to -\Phi_2$ part, and a doublet mixing term of the form

$$\mathscr{L}_{\text{mix}} = -m_{12}^2 \left(\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1 \right) . \tag{6.6}$$

Remark that for the potential (6.4) the couplings m_1^2, m_2^2 can be taken to be negative, so that one can define a non-trivial vacuum expectation value. Such a vacuum expectation value can be writen in the form

$$\langle\phi_3\rangle = v_1 \quad , \quad \langle\phi_7\rangle = v_2 \ , \tag{6.7}$$
$$\langle\phi_1\rangle = \langle\phi_2\rangle = \langle\phi_4\rangle = \langle\phi_5\rangle = \langle\phi_6\rangle = \langle\phi_8\rangle = 0 \ .$$

Note that this form is uniquely defined up to a $U(1)\times SU(2)$ transformation. The vacuum expectation values satisfy

$$\begin{cases} \left(m_1^2 + \frac{\kappa_1}{2}v_1^2 + \kappa_T v_2^2\right) v_1 = m_{12}^2 v_2 \\ \left(m_2^2 + \frac{\kappa_2}{2}v_2^2 + \kappa_T v_1^2\right) v_2 = m_{12}^2 v_1 \end{cases} , \tag{6.8a}$$

where $\kappa_T \equiv (\kappa_3+\kappa_4+\kappa_5)$. We are then able to express the system in terms of fluctuations around this vacuum. The mass matrix in this basis satisfies

$$M_{ij} = \frac{1}{2}\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\bigg|_{\phi_3=v_1\ ,\ \phi_7=v_2} , \tag{6.9}$$

In our model, the mass-matrix is given by

$$M_{(ij)} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} , \tag{6.10}$$

with

$$N_{11} = m_{12}^2 \frac{v_2}{v_1} \mathbb{I}_{4\times 4} - \begin{pmatrix} v_2^2(\kappa_4+\kappa_5) & 0 & 0 & 0 \\ 0 & v_2^2(\kappa_4+\kappa_5) & 0 & 0 \\ 0 & 0 & -\kappa_1 v_1^2 & 0 \\ 0 & 0 & 0 & \kappa_5 v_2^2 \end{pmatrix} , \tag{6.11a}$$

$$N_{12} = N_{21} = v_1 v_2 \begin{pmatrix} \kappa_4+\kappa_5 & 0 & 0 & 0 \\ 0 & \kappa_4+\kappa_5 & 0 & 0 \\ 0 & 0 & 2\kappa_T & 0 \\ 0 & 0 & 0 & 2\kappa_5 \end{pmatrix} - m_{12}^2 \mathbb{I}_{4\times 4} , \tag{6.11b}$$

$$N_{22} = m_{12}^2 \frac{v_2}{v_1} \mathbb{I}_{4\times 4} - \begin{pmatrix} v_1^2(\kappa_4+\kappa_5) & 0 & 0 & 0 \\ 0 & v_1^2(\kappa_4+\kappa_5) & 0 & 0 \\ 0 & 0 & -\kappa_2 v_2^2 & 0 \\ 0 & 0 & 0 & \kappa_5 v_1^2 \end{pmatrix} . \tag{6.11c}$$

In order to define the eigenvectors, it is convenient to express the doublets as

$$\Phi_1 = \begin{pmatrix} \phi_1^+ \\ v_1 + \rho_1 + i\psi_1 \end{pmatrix} , \quad \Phi_2 = \begin{pmatrix} \phi_2^+ \\ v_2 + \rho_2 + i\psi_2 \end{pmatrix} , \tag{6.12}$$

where ϕ_i^+ are complex scalar fields and we notate $\left(\phi_i^+\right)^* = \phi_i^-$ and ρ_i and ψ_i are real fields. We use this to express the mass eigenstates as

$$\begin{pmatrix} H \\ h \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} , \tag{6.13a}$$

$$\begin{pmatrix} G^+ \\ H^+ \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix} , \tag{6.13b}$$

$$\begin{pmatrix} G \\ D \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} . \tag{6.13c}$$

Here α is the mixing angle between the neutral states ρ_1 and ρ_2, while β is the angle between the charged states ϕ_1^+, ϕ_2^+. We will discuss these angles and their values in more detail, when we compare this model to a non-Hermitian two-Higgs-doublet model. The system has three massless Goldstone fields G, G^+, G^-, while the other fields all have non-zero masses. From this, we can see that after spontaneous symmetry breaking of our system, we end up with two charged Higgs bosons H^+, H^- and three neutral Higgs bosons H, h, D.

6.2 Non-Hermitian Two-Higgs Doublet Model

In the previous section we discussed the Hermitian 2HDM system where the Higgs potential was given by equation (6.4). We have already formulated a $\mathscr{P}\mathscr{T}$-symmetric Quantum Field Theory Lagrangian of the form

$$\mathscr{L} = \partial^\alpha \phi_1^* \partial^\alpha \phi_1 + \partial^\alpha \phi_2^* \partial_\alpha \phi_2 - m_1^2 |\phi_1|^2 - m_2^2 |\phi_2|^2 - \mu^2 \left(\phi_1^* \phi_2 - \phi_2^* \phi_1\right) - U_{\text{int}} \,, \tag{6.14}$$

that was later made gauge invariant under an Abelian gauge symmetry. We saw that in order to obtain consistent equations of motion for this system, the gauge field should couple to a non-conserved current. We want to further develop the formulation of $\mathscr{P}\mathscr{T}$-symmetric gauge theories by including a non-Abelian gauge symmetry and Kibble's non-Abelian generalisation [85] of the Englert-Brout-Higgs mechanism.

We study a minimal extension of the model that was previously discussed. This model contains two complex scalar doublets and possesses a $U(1) \times SU(2)$ gauge symmetry as the Standard Model. In this way we naturally develop a non-Hermitian two-Higgs-doublet model similar to the one discussed in equation (6.4) but with an anti-Hermitian doublet mixing term of the form $\mu^2 \left(\Phi_1^\dagger \Phi_2 - \Phi_2^\dagger \Phi_1\right)$ instead of the Hermitian doublet mixing term $m_{12}^2 \left(\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1\right)$ in equation (6.4).

We show for this system how the gauge should be fixed consistently by using Becchi-Rouet-Stora-Tyutin (BRST) invariance [86, 87]. After this, we can obtain consistent equations of motion and derive the spectrum of our theory. We then discuss the eigenmasses of this system and compare those to a Hermitian model, similar to the system with potential (6.4) in the limit $\kappa_1 \to \kappa, \quad \kappa_2, \kappa_3, \kappa_4, \kappa_5 \to 0$. We show that there is a significant difference between the two models, thus, this non-Hermitian model offers experimental prospects that might be discussed in later works.

6.2.1 Scalar Lagrangian

We start from a Lagrangian of the form

$$\begin{aligned}\mathscr{L} =& \partial_\alpha \Phi_1^\dagger \partial^\alpha \Phi_1 + \partial_\alpha \Phi_2^\dagger \partial^\alpha \Phi_2 - m_1^2 |\Phi_1|^2 - m_2^2 |\Phi_2|^2 \\ & - \mu^2 \left(\Phi_1^\dagger \Phi_2 - \Phi_2^\dagger \Phi_1 \right) - \frac{\kappa}{4} |\Phi_1|^4 \ ,\end{aligned} \tag{6.15}$$

where Φ_i are complex doublets of the form

$$\Phi_i = \begin{pmatrix} \phi_{ia} \\ \phi_{ib} \end{pmatrix} , \qquad i = 1, 2 \ . \tag{6.16}$$

The μ^2 terms in this Lagrangian correspond to the anti-Hermitian contribution to our theory. We will follow a similar method to gauge this system as previously outlined in Chapter 4.

6.2.1.1 Eigenvalues

This system (6.15) is invariant under the $\mathscr{PT}$-symmetry, acting on the c-number fields as

$$\begin{aligned}\mathscr{PT}: \qquad & \Phi_1(t,x) \to \Phi_1'(-t,-x) = \Phi_1^*(t,x) \ , \\ & \Phi_2(t,x) \to \Phi_2'(-t,-x) = -\Phi_2^*(t,x) \ ,\end{aligned} \tag{6.17}$$

under which Φ_1 acts as a scalar doublet whereas Φ_2 acts as a pseudoscalar doublet. The eigenvalues of the squared mass matrix

$$M_\pm^2 = \frac{m_1^2 + m_2^2}{2} \pm \frac{1}{2} \sqrt{(m_1^2 - m_2^2)^2 - 4\mu^4} \ , \tag{6.18}$$

are real provided the following inequality holds:

$$2|\mu^2| \leq |m_1^2 - m_2^2| \ . \tag{6.19}$$

We assume for now that $m_1^2, m_2^2 > 0$ until we derive non-trivial vacuum expectation values. Until then the condition (6.19) should be satisfied for the energies to be real. Note that the eigenvalues become degenerate at $|\mu^2| = |m_1^2 - m_2^2|/2$. This marks the exceptional point, which lies at the boundary between the regions of unbroken and broken $\mathscr{PT}$ symmetry. At this point, the squared mass matrix becomes defective and we lose an eigendirection. We will discuss the exceptional points in more detail in Section 6.2.3.4. Similar as discussed previously, the equations of motion are obtained by varying the action with respect to Φ_i or to $\Phi_i^\dagger$, i.e.

$$\frac{\delta S}{\delta \Phi_i^\dagger} \equiv \frac{\partial \mathscr{L}}{\partial \Phi_i^\dagger} - \partial_\alpha \frac{\partial \mathscr{L}}{\partial\left(\partial_\alpha \Phi_i^\dagger\right)} = 0 \qquad \text{or} \qquad \frac{\delta S}{\delta \Phi_i} \equiv \frac{\partial \mathscr{L}}{\partial \Phi_i} - \partial_\alpha \frac{\partial \mathscr{L}}{\partial\left(\partial_\alpha \Phi_i\right)} = 0 \ . \tag{6.20}$$

Because the non-Hermitian mass term is proportional to μ^2, these different equations of motion are not equivalent for non-trivial solutions. As discussed before, since the eigenvalues (6.18) depend on μ^4 only, we can see that these two sets of equations of motion are physically equivalent. In this work, we choose here the equations of motion provided by the variation of the action with respect to $\Phi_i^\dagger$:

$$0 = \Box\Phi_1 + m_1^2\Phi_1 + \mu^2\Phi_2 + \frac{\kappa}{2}|\Phi_1|^2\Phi_1 \ , \tag{6.21a}$$

$$0 = \Box\Phi_2 + m_2^2\Phi_2 - \mu^2\Phi_1 \ , \tag{6.21b}$$

together with their Hermitian conjugates

$$0 = \Box\Phi_1^\dagger + m_1^2\Phi_1^\dagger + \mu^2\Phi_2^\dagger + \frac{\kappa}{2}|\Phi_1|^2\Phi_1^\dagger \ , \tag{6.22a}$$

$$0 = \Box\Phi_2^\dagger + m_2^2\Phi_2^\dagger - \mu^2\Phi_1^\dagger \ . \tag{6.22b}$$

As discussed before, this formulation differs from that suggested in [58], where the author introduces a similarity transformation that transforms the non-Hermitian

Lagrangian $\mathscr{L}$ to a Hermitian one $\mathscr{L}'$. The difference in approach will be reflected in differences in the masses of the gauge fields, which we already partially discussed in Chapter 4.

6.2.1.2 Conserved Currents

The Lagrangian (6.15) is invariant under the global $U(1)$ transformations

$$\Phi_1 \quad \rightarrow \quad e^{-i\frac{g'}{2}\beta_0}\Phi_1 \ , \tag{6.23a}$$

$$\Phi_2 \quad \rightarrow \quad e^{-i\frac{g'}{2}\beta_0}\Phi_2 \ , \tag{6.23b}$$

which correspond to the current

$$I_+^\alpha = i\frac{g'}{2}\left(\left[\Phi_1^\dagger\left(\partial^\alpha\Phi_1\right) - \left(\partial^\alpha\Phi_1^\dagger\right)\Phi_1\right] + \left[\Phi_2^\dagger\left(\partial^\alpha\Phi_2\right) - \left(\partial^\alpha\Phi_2^\dagger\right)\Phi_2\right]\right) \ , \tag{6.24}$$

and also invariant under the $SU(2)$ transformations

$$\Phi_1 \quad \rightarrow \quad e^{-i\frac{g}{2}\vec{\beta}\cdot\vec{\tau}}\Phi_1 \ , \tag{6.25a}$$

$$\Phi_2 \quad \rightarrow \quad e^{-i\frac{g}{2}\vec{\beta}\cdot\vec{\tau}}\Phi_2 \ , \tag{6.25b}$$

which correspond to the current

$$\vec{J}_+^\alpha = i\frac{g}{2}\left(\left[\Phi_1^\dagger\vec{\tau}\left(\partial^\alpha\Phi_1\right) - \left(\partial^\alpha\Phi_1^\dagger\right)\vec{\tau}\Phi_1\right] + \left[\Phi_2^\dagger\vec{\tau}\left(\partial^\alpha\Phi_2\right) - \left(\partial^\alpha\Phi_2^\dagger\right)\vec{\tau}\Phi_2\right]\right) \ , \tag{6.26}$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ is composed of the Pauli matrices.

As is the case for a general non-Hemritian system, the equations of motion (6.21) show that these currents are not conserved:

$$\partial_\alpha I_+^\alpha = ig'\mu^2\left(\Phi_2^\dagger\Phi_1 - \Phi_1^\dagger\Phi_2\right) \ , \tag{6.27a}$$

$$\partial_\alpha \vec{J}_+^\alpha = ig\mu^2\left(\Phi_2^\dagger\vec{\tau}\Phi_1 - \Phi_1^\dagger\vec{\tau}\Phi_2\right) \ , \tag{6.27b}$$

except at the Hermitian point $\mu^2 = 0$. In this model, the conserved currents are, in fact,

$$I_-^\alpha = i\frac{g'}{2}\left(\left[\Phi_1^\dagger\left(\partial^\alpha\Phi_1\right) - \left(\partial^\alpha\Phi_1^\dagger\right)\Phi_1\right] - \left[\Phi_2^\dagger\left(\partial^\alpha\Phi_2\right) - \left(\partial^\alpha\Phi_2^\dagger\right)\Phi_2\right]\right)\ , \tag{6.28a}$$

$$\vec{J}_-^\alpha = i\frac{g}{2}\left(\left[\Phi_1^\dagger\vec{\tau}\left(\partial^\alpha\Phi_1\right) - \left(\partial^\alpha\Phi_1^\dagger\right)\vec{\tau}\Phi_1\right] - \left[\Phi_2^\dagger\vec{\tau}\left(\partial^\alpha\Phi_2\right) - \left(\partial^\alpha\Phi_2^\dagger\right)\vec{\tau}\Phi_2\right]\right)\ , \tag{6.28b}$$

which correspond to the following transformations:

$$\Phi_1 \rightarrow e^{-i\frac{g'}{2}\beta_0}\Phi_1\ , \tag{6.29a}$$

$$\Phi_2 \rightarrow e^{+i\frac{g'}{2}\beta_0}\Phi_2\ , \tag{6.29b}$$

and

$$\Phi_1 \rightarrow e^{-i\frac{g}{2}\vec{\beta}\cdot\vec{\tau}}\Phi_1\ , \tag{6.30a}$$

$$\Phi_2 \rightarrow e^{+i\frac{g}{2}\vec{\beta}\cdot\vec{\tau}}\Phi_2\ . \tag{6.30b}$$

The relative sign between the charge assignments of the two fields reflects the usual interpretation of viable $\mathscr{P}\mathscr{T}$-symmetric theories as systems with coupled gain and loss.

6.2.2 Gauging the Scalar Model

Since the conserved currents do not correspond to the usual Noether currents, gauging the model (6.15) is again a non-trivial matter. In order to do this, we will need to include the covariant derivatives.

6.2.2.1 Coupling to the Noether Currents

We introduce an Abelian $U(1)$ gauge field B^α and an $SU(2)$ gauge field $\vec{W}^\alpha$, together to define a $U(1)\times SU(2)$ gauge transformations of the form

$$\Phi_i \to e^{-i\frac{g'}{2}\beta_0}e^{-i\frac{g}{2}\vec{\beta}\cdot\vec{\tau}}\Phi_i\ , \tag{6.31a}$$

$$\vec{W}^\alpha \to \vec{W}^\alpha + g\left(\vec{\beta}\times\vec{W}^\alpha\right) + \partial^\alpha\vec{\beta} = \vec{W}^\alpha + \mathscr{D}^\alpha\vec{\beta}\ , \tag{6.31b}$$

$$B^\alpha \to B^\alpha + \partial^\alpha\beta_0\ , \tag{6.31c}$$

here $\mathscr{D}^\alpha\vec{\beta} \equiv \partial^\alpha\vec{\beta} - g(\vec{W}^\alpha\times\vec{\beta})$. Taking into account the results we obtained for the Abilean gauge field in Chapter 4, the gauge fields should couple to the currents I_+^α and $\vec{J}_+^\alpha$, such that the scalar kinetic terms are given by

$$\begin{aligned}\mathscr{L}_{\text{kin}} =& [D_\alpha\Phi_1]^\dagger D^\alpha\Phi_1 + [D_\alpha\Phi_2]^\dagger D^\alpha\Phi_2 \\ =& \partial_\alpha\Phi_1^\dagger\partial^\alpha\Phi_1 + \partial_\alpha\Phi_2^\dagger\partial^\alpha\Phi_2 + \frac{i}{2}\partial_\alpha\Phi_1^\dagger\left(g'B^\alpha + g\vec{\tau}\cdot\vec{W}^\alpha\right)\Phi_1\,, \\ &-\frac{i}{2}\Phi_1^\dagger\left(g'B^\alpha + g\vec{\tau}\cdot\vec{W}^\alpha\right)\partial_\alpha\Phi_1 + \frac{i}{2}\partial_\alpha\Phi_2^\dagger\left(g'B^\alpha + g\vec{\tau}\cdot\vec{W}^\alpha\right)\Phi_2\,, \\ &-\frac{i}{2}\Phi_2^\dagger\left(g'B^\alpha + g\vec{\tau}\cdot\vec{W}^\alpha\right)\partial_\alpha\Phi_2 + \frac{1}{4}\Phi_1^\dagger\left(g'B + g\vec{\tau}\cdot\vec{W}\right)^2\Phi_1 \\ &+\frac{1}{4}\Phi_2^\dagger\left(g'B + g\vec{\tau}\cdot\vec{W}\right)^2\Phi_2\ ,\end{aligned} \tag{6.32}$$

where D_α is given by the usual minimal-coupling prescription, i.e.,

$$D^\alpha\Phi_i = \partial^\alpha\Phi_i + \frac{ig'}{2}B^\alpha\Phi_i + \frac{ig}{2}\left[\vec{\tau}\cdot\vec{W}^\alpha\right]\Phi_i\ . \tag{6.33}$$

Similarly as one does in the Standard Model, one can now introduce new gauge field variables by rotating the fields as

$$B^\alpha = \cos\theta_{\rm W}A^\alpha - \sin\theta_{\rm W}Z^\alpha\ , \tag{6.34}$$

$$W_1^\alpha = \frac{W^\alpha + W^{\alpha\dagger}}{\sqrt{2}}\ ,\qquad W_3^\alpha = \sin\theta_{\rm W}A^\alpha + \cos\theta_{\rm W}Z^\alpha\ ,\qquad W_2^\alpha = i\frac{W^\alpha - W^{\alpha\dagger}}{\sqrt{2}}\ ,$$

where $\theta_{\rm W}$ is the weak mixing angle [88], in order to write the kinetic part of the Lagrangian (6.32) as

$$\begin{aligned}
\mathscr{L}_{\rm kin} =& \partial_\alpha \Phi_1^\dagger \partial^\alpha \Phi_1 + \partial_\alpha \Phi_2^\dagger \partial^\alpha \Phi_2 \\
& - W_\alpha \left[\frac{J^\alpha_{+,1} + iJ^\alpha_{+,2}}{\sqrt{2}} \right] - W^\dagger_\alpha \left[\frac{J^\alpha_{+,1} - iJ^\alpha_{+,2}}{\sqrt{2}} \right] \\
& - Z_\alpha \left[J^\alpha_{+,3} \cos\theta_{\rm W} - I^\alpha_+ \sin\theta_{\rm W} \right] - A_\alpha \left[J^\alpha_{+,3} \sin\theta_{\rm W} + I^\alpha_+ \cos\theta_{\rm W} \right] \\
& + \frac{g^2}{2} W^\dagger_\alpha W^\alpha \left(|\Phi_1|^2 + |\Phi_2|^2 \right) \\
& + \frac{1}{4} Z_\alpha Z^\alpha \sum_i \Phi_i^\dagger \Big(\left[g'^2 \sin^2\theta_{\rm W} + g^2 \cos^2\theta_{\rm W} \right] \mathbb{I} - 2gg' \cos\theta_{\rm W} \sin\theta_{\rm W} \tau_3 \Big) \Phi_i \\
& + \frac{1}{4} A_\alpha A^\alpha \sum_i \Phi_i^\dagger \Big(\left[g'^2 \cos^2\theta_{\rm W} + g^2 \sin^2\theta_{\rm W} \right] \mathbb{I} + 2gg' \cos\theta_{\rm W} \sin\theta_{\rm W} \tau_3 \Big) \Phi_i \\
& + \frac{1}{4} Z_\alpha A^\alpha \sum_i \Phi_i^\dagger \Big(\left[(g^2 - g'^2) \sin 2\theta_{\rm W} \mathbb{I} + 2gg' \cos 2\theta_{\rm W} \tau_3 \right] \Big) \Phi_i \\
& + \frac{1}{2} gg' \Big(\cos\theta_{\rm W} A^\alpha - \sin\theta_{\rm W} Z^\alpha \Big) \sum_i \Phi_i^\dagger \left(W_\alpha \left[\frac{\tau_1 + i\tau_2}{\sqrt{2}} \right] + W^\dagger_\alpha \left[\frac{\tau_1 - i\tau_2}{\sqrt{2}} \right] \right) \Phi_i \, .
\end{aligned} \tag{6.35}$$

For the kinetic part of the gauge fields, the Lagrangian is given by

$$\begin{aligned}
\mathscr{L}_{\rm gauge} =& -\frac{1}{4} \vec{W'}_{\alpha\beta} \cdot \vec{W'}^{\alpha\beta} - \frac{1}{4} B_{\alpha\beta} B^{\alpha\beta} \\
=& -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{4} Z_{\alpha\beta} Z^{\alpha\beta} - \frac{1}{2} W^\dagger_{\alpha\beta} W^{\alpha\beta} \ ,
\end{aligned} \tag{6.36}$$

with

$$\vec{W'}_{\alpha\beta} = \partial_\beta \vec{W}_\alpha - \partial_\alpha \vec{W}_\beta + g \left(\vec{W}_\alpha \times \vec{W}_\beta \right) \ , \tag{6.37a}$$

$$B_{\alpha\beta} = \partial_\beta B_\alpha - \partial_\alpha B_\beta \ , \tag{6.37b}$$

$$\begin{aligned}
W_{\alpha\beta} =& \left[\partial_\beta + ig \left(\sin\theta_{\rm W} A_\beta + \cos\theta_{\rm W} Z_\beta \right) \right] W_\alpha \\
& - \left[\partial_\alpha + ig \left(\sin\theta_{\rm W} A_\alpha + \cos\theta_{\rm W} Z_\alpha \right) \right] W_\beta \ ,
\end{aligned} \tag{6.37c}$$

$$F_{\alpha\beta} = \partial_\beta A_\alpha - \partial_\alpha A_\beta + ig \sin\theta_{\rm W} \left[W^\dagger_\alpha W_\beta - W^\dagger_\beta W_\alpha \right] \ , \tag{6.37d}$$

$$Z_{\alpha\beta} = \partial_\beta Z_\alpha - \partial_\alpha Z_\beta + ig \cos\theta_{\rm W} \left[W^\dagger_\alpha W_\beta - W^\dagger_\beta W_\alpha \right] \ . \tag{6.37e}$$

6.2.2.2 Consistent Field Equations

The gauge fields couple to currents that are non-conserved. As was the case for the Abelian symetry, aditional terms need to be added to have consistent Maxwell equations. To achieve this, it should be enough to add the usual gauge fixing terms for non-Abelian gauge theories

$$\begin{aligned}\mathscr{L}_{\mathrm{GF}} &= \partial_\alpha \vec{\bar{\eta}} \cdot [\mathscr{D}^\alpha \vec{\eta}] - \frac{1}{2\xi}\left[(\partial_\alpha B^\alpha)^2 + |\partial_\alpha \vec{W}^\alpha|^2\right] \\ &= \partial_\alpha \bar{\chi}^\dagger \Big([\partial^\alpha + ig(\sin\theta_{\mathrm{W}} A^\alpha + \cos\theta_{\mathrm{W}} Z^\alpha)]\,\chi - igW^\alpha \eta_3 \Big) \\ &+ \partial_\alpha \bar{\chi} \Big([\partial^\alpha - ig(\sin\theta_{\mathrm{W}} A^\alpha + \cos\theta_{\mathrm{W}} Z^\alpha])\,\chi^\dagger + igW^{\alpha\dagger}\eta_3 \Big) \\ &+ \partial_\alpha \bar{\eta}_3 \left(\partial^\alpha \eta_3 + ig\left[W^\alpha \chi^\dagger - W^{\alpha\dagger}\chi \right)\right] \\ &- \frac{1}{2\xi}\left[(\partial_\alpha A^\alpha)^2 + (\partial_\alpha Z^\alpha)^2 + 2|\partial_\alpha W^\alpha|^2\right] ,\end{aligned} \tag{6.38}$$

where $\vec{\eta}$ and $\vec{\bar{\eta}}$ are the ghost fields and

$$\bar{\chi} \equiv \frac{\bar{\eta}_1 - i\bar{\eta}_2}{\sqrt{2}} , \qquad \chi \equiv \frac{\eta_1 - i\eta_2}{\sqrt{2}} . \tag{6.39}$$

Remark that in this case these terms need to be added to the classical equations of motion and not just at the Quantum level to consistently define the path integral . The equations of motion for the full Lagrangian are then given by

$$0 = D_\alpha D^\alpha \Phi_1 + m_1^2 \Phi_1 + \mu^2 \Phi_2 + \frac{\kappa}{2}|\Phi_1|^2 \Phi_1 , \tag{6.40a}$$

$$0 = D_\alpha D^\alpha \Phi_2 + m_2^2 \Phi_2 - \mu^2 \Phi_1 , \tag{6.40b}$$

$$0 = \mathscr{D}_\beta \vec{W'}^{\beta\alpha} + \vec{\mathscr{J}}_+^\alpha - \frac{1}{\xi}\partial^\alpha \partial^\beta \vec{W}_\beta - g\left(\partial^\alpha \vec{\bar{\eta}} \times \vec{\eta}\right) , \tag{6.40c}$$

$$0 = \partial_\beta B^{\beta\alpha} + \mathscr{I}_+^\alpha - \frac{1}{\xi}\partial^\alpha \partial^\beta B_\beta , \tag{6.40d}$$

$$0 = \partial_\alpha \mathscr{D}^\alpha \vec{\eta} , \tag{6.40e}$$

$$0 = \mathscr{D}_\alpha \partial^\alpha \vec{\bar{\eta}} , \tag{6.40f}$$

together with their Hermitian conjugates, where we define

$$\mathscr{I}_+^\alpha \equiv i\frac{g'}{2}\left(\left[\Phi_1^\dagger(D^\alpha\Phi_1) - (D^\alpha\Phi_1)^\dagger\,\Phi_1\right] + \left[\Phi_2^\dagger(D^\alpha\Phi_2) - (D^\alpha\Phi_2)^\dagger\,\Phi_2\right]\right)\,, \tag{6.41a}$$

$$\vec{\mathscr{J}}_+^\alpha \equiv i\frac{g}{2}\left(\left[\Phi_1^\dagger\vec{\tau}(D^\alpha\Phi_1) - (D^\alpha\Phi_1)^\dagger\,\vec{\tau}\Phi_1\right] + \left[\Phi_2^\dagger\vec{\tau}(D^\alpha\Phi_2) - (D^\alpha\Phi_2)^\dagger\,\vec{\tau}\Phi_2\right]\right)\,. \tag{6.41b}$$

From the equations of motion (6.40) one can derive the current divergence, which leads to the constraints

$$\frac{1}{\xi}\mathscr{D}_\alpha\partial^\alpha\partial^\beta\vec{W}_\beta = ig\mu^2\left(\Phi_2^\dagger\vec{\tau}\Phi_1 - \Phi_1^\dagger\vec{\tau}\Phi_2\right) - g\partial^\alpha\vec{\bar{\eta}}\times\mathscr{D}_\alpha\vec{\eta}\,, \tag{6.42a}$$

$$\frac{1}{\xi}\Box\partial^\beta B_\beta = ig'\mu^2\left(\Phi_2^\dagger\Phi_1 - \Phi_1^\dagger\Phi_2\right)\,, \tag{6.42b}$$

which must be satisfied in order for the field equations to be consistent. We will later show that the BRST symmetry allows one to write the latter constraints independently of the ghost fields, as

$$\frac{1}{\xi}\mathscr{D}_\alpha\partial^\alpha\partial^\beta\vec{W}_\beta = \frac{ig\mu^2}{2}\left(\Phi_2^\dagger\vec{\tau}\Phi_1 - \Phi_1^\dagger\vec{\tau}\Phi_2\right)\,, \tag{6.43a}$$

$$\frac{1}{\xi}\Box\partial^\beta B_\beta = ig'\mu^2\left(\Phi_2^\dagger\Phi_1 - \Phi_1^\dagger\Phi_2\right)\,. \tag{6.43b}$$

We can summarise our approach as follows. In order to respect gauge invariance, we need to couple the gauge fields to non-conserved currents. However, in order to still have consistent Maxwell equations, we need to introduce gauge-fixing terms, which restrict gauge invariance, but imply consistent field equations. The residual gauge invariance is enough to ensure that gauge fields remain massless in the absence of spontaneous symmetry breaking (SSB), and it is defined by the gauge functions $\beta_0, \vec{\beta}$ satisfying

$$\partial_\alpha\mathscr{D}^\alpha\vec{\beta} = 0\,, \tag{6.44a}$$

$$\Box\beta_0 = 0\,. \tag{6.44b}$$

We therefore obtain a consistent gauge theory with a non-Hermitian scalar sector, as we did in the Abelian case.

6.2.2.3 BRST Transformation

The constraints in equation (6.43) for $\vec{W}_\alpha$ are derived form the BRST symmetries of our gauge fixed Lagrangian. This symmetry is defined after introducing the auxiliary field $\vec{T}$ to rewrite the gauge-fixing Lagrangian (6.38) in the alternative form

$$\mathscr{L}_{\mathrm{GF}} = \partial_\alpha \vec{\bar{\eta}} \cdot \mathscr{D}^\alpha \vec{\eta} + \frac{\xi}{2}|\vec{T}|^2 - \vec{T} \cdot \partial^\alpha \vec{W}_\alpha - \frac{1}{2\xi}\left(\partial_\alpha B^\alpha\right)^2 \ , \tag{6.45}$$

and the original Lagrangian (6.38) can be recovered after integrating out $\vec{T}$. The BRST transformations are defined as

$$\begin{aligned}
\delta\phi_i &= -i\frac{g}{2}\theta\left(\vec{\tau}\cdot\vec{\eta}\right)\phi_i \ , && (6.46\text{a})\\
\delta\vec{W}^\alpha &= \theta\mathscr{D}^\alpha\vec{\eta} \ , && (6.46\text{b})\\
\delta B^\alpha &= 0 \ , && (6.46\text{c})\\
\delta\vec{\bar{\eta}} &= -\theta\vec{T} \ , && (6.46\text{d})\\
\delta\vec{\eta} &= \frac{g}{2}\theta\left(\vec{\eta}\times\vec{\eta}\right) \ , && (6.46\text{e})\\
\delta\vec{T} &= 0 \ , && (6.46\text{f})
\end{aligned}$$

where θ is an infinitesimal Grassmann parameter. The gauge-invariant terms (6.32) and (6.36) in the Lagrangian are trivial invariant under the BRST transformation, and the gauge-fixing Lagrangian (6.45) transforms as a total derivative, so the action is invariant under this BRST transformation. Using the auxiliary field $\vec{T}$, the equation of motion (6.40c) for the gauge field $\vec{W}^\alpha$ can be written in the form

$$0 = \mathscr{D}_\beta {\vec{W}'}^{\beta\alpha} + \vec{\mathscr{J}}_+^\alpha - \partial^\alpha\vec{T} - g\left(\partial^\alpha\vec{\bar{\eta}}\times\vec{\eta}\right) \ , \tag{6.47}$$

and a covariant derivative leads to

$$\mathscr{D}_\alpha\partial^\alpha\vec{T} = ig\mu^2\left(\Phi_2^\dagger\vec{\tau}\Phi_1 - \Phi_1^\dagger\vec{\tau}\Phi_2\right) - g\partial^\alpha\vec{\bar{\eta}}\times\mathscr{D}_\alpha\vec{\eta} \ . \tag{6.48}$$

A BRST transformation of equation (6.40f) then leads to the relation

$$0 = \delta\left(\mathscr{D}_\alpha \partial^\alpha \vec{\bar{\eta}}\right) = -\theta\left(\mathscr{D}_\alpha \partial^\alpha \vec{T} - g\partial_\alpha \vec{\bar{\eta}} \times \mathscr{D}^\alpha \vec{\eta}\right) , \tag{6.49}$$

so that

$$\mathscr{D}_\alpha \partial^\alpha \vec{T} = g\partial_\alpha \vec{\bar{\eta}} \times \mathscr{D}^\alpha \vec{\eta} , \tag{6.50}$$

which, together with equation (6.48), leads to

$$\mathscr{D}_\alpha \partial^\alpha \vec{T} = \frac{ig\mu^2}{2}\left(\Phi_2^\dagger \vec{\tau} \Phi_1 - \Phi_1^\dagger \vec{\tau} \Phi_2\right) . \tag{6.51}$$

Since, from the equations of motion for $\vec{T}$, one finds

$$\vec{T} = \frac{1}{\xi}\partial_\alpha \vec{W}^\alpha , \tag{6.52}$$

one finally obtains the expected constraint

$$\frac{1}{\xi}\mathscr{D}_\alpha \partial^\alpha \partial^\beta \vec{W}_\beta = \frac{ig\mu^2}{2}\left(\Phi_2^\dagger \vec{\tau} \Phi_1 - \Phi_1^\dagger \vec{\tau} \Phi_2\right) , \tag{6.53}$$

which, unlike equation (6.42a), is independent of the ghost fields. For further discussions of BRST (and anti-BRST) symmetries in the context of non-Hermitian field theories, see [89].

6.2.3 Spontaneous Symmetry Breaking

The Lagrangian we discussed in equation (6.15) does not have a non-trivial vacuum expectation value since the masses $m_1^2, m_2^2 > 0$. In what follows, we change the sign of m_1^2 to allow for spontaneous symmetry breaking (SSB) and discuss the expectation values and vector masses.

6.2.3.1 Vacuum Expectation Value

With this change of sign, the Lagrangian (6.15) has a symmetry-breaking vacuum that is given by

$$\frac{\kappa}{2}|\langle\Phi_1\rangle|^2 = m_1^2 - \frac{\mu^4}{m_2^2} \,, \tag{6.54a}$$

$$\langle\Phi_2\rangle = \frac{\mu^2}{m_2^2}\langle\Phi_1\rangle \,, \tag{6.54b}$$

which, according to equation (6.54a), is physical as long as

$$m_1^2 m_2^2 > \mu^4 \,. \tag{6.55}$$

Remark that the condition (6.19) no longer applies here, since the sign of m_1^2 has changed. The spectrum of this theory must be determined after symmetry breaking, and thus the condition that ensures the reality of eigenmasses will be determined then.

The vacuum is defined up to a $U(1)\times SU(2)$ transformation, and it is again choosen to be of the form

$$\langle\Phi_1\rangle = \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \equiv V_1 \,, \qquad \langle\Phi_2\rangle = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \equiv V_2 \,, \tag{6.56}$$

where

$$v_1 = \sqrt{\frac{2}{\kappa}\left(m_1^2 - \frac{\mu^4}{m_2^2}\right)} \,, \qquad v_2 = \frac{\mu^2}{m_2^2}\sqrt{\frac{2}{\kappa}\left(m_1^2 - \frac{\mu^4}{m_2^2}\right)} \,. \tag{6.57}$$

With this choice, the vacuum expectation value is unbroken by the transformation

$$\langle\Phi_i\rangle \to e^{-i\frac{e}{2}\beta_0(\mathbb{I}+\tau_3)}\langle\Phi_i\rangle = \begin{pmatrix} e^{-ie\beta_0} & 0 \\ 0 & 1 \end{pmatrix}\langle\Phi_i\rangle = \langle\Phi_i\rangle \,, \tag{6.58}$$

such that the Abelian subgroup of $U(1)\times SU(2)$ generated by $\sigma = \mathbb{I}+\tau_3$ remains unbroken. This subgroup corresponds to the electromagnetic interaction, with Noether current

$$
\begin{aligned}
Q^\alpha =& \frac{ie}{2}\left[\Phi_1^\dagger\sigma(\partial^\alpha\Phi_1)-(\partial_1^\alpha\Phi_1^\dagger)\sigma\Phi_1\right]+\frac{ie}{2}\left[\Phi_2^\dagger\sigma(\partial^\alpha\Phi_2)-(\partial_2^\alpha\Phi_2^\dagger)\sigma\Phi_2\right]\\
=&\frac{e}{g'}I_+^\alpha+\frac{e}{g}J_{+,3}^\alpha\ .
\end{aligned}
\tag{6.59}
$$

From equation (6.35), we see that the gauge field A^μ should couple to the current $I_+^\alpha\cos\theta_\mathrm{W}+J_{+,3}^\alpha\sin\theta_\mathrm{W}$, which can be identified with the current (6.59) if

$$
e=g'\cos\theta_\mathrm{W}=g\sin\theta_\mathrm{W}\ . \tag{6.60}
$$

The $\mathrm{U}(1)_\mathrm{EM}$ charge is conserved at the tree-level, although the Noether current is generally not conserved. Exploration of the possibility of charge non-conservation beyond the tree level lies beyond the scope of this work. Its existence and observability would in principle depend upon the completion of the bosonic model considered here to include fermions, which is also a topic for future work. We can then express the scalar Lagrangian in terms of fluctuations around the vacuum (6.56) as

$$
\begin{aligned}
\mathscr{L}_\mathrm{scal} =&\partial_\alpha\hat\Phi_1^\dagger\partial^\alpha\hat\Phi_1+\partial_\alpha\hat\Phi_2^\dagger\partial^\alpha\hat\Phi_2+\frac{2\mu^4}{m_2^2}\left(V_1^\dagger\hat\Phi_1\right)-2m_2^2\left(V_2^\dagger\hat\Phi_2\right)\\
&-m_2^2|\hat\Phi_2|^2+\frac{\mu^4}{m_2^2}|\hat\Phi_1|^2-\frac{\kappa}{4}\left(V_1^\dagger\hat\Phi_1+\hat\Phi_1^\dagger V_1\right)^2-\mu^2\left(\hat\Phi_1^\dagger\hat\Phi_2-\hat\Phi_2^\dagger\hat\Phi_1\right)\\
&-\frac{\kappa}{2}\left(V_1^\dagger\hat\Phi_1+\hat\Phi_1^\dagger V_1\right)|\hat\Phi_1|^2-\frac{\kappa}{4}|\hat\Phi_1|^4\ ,
\end{aligned}
\tag{6.61}
$$

where

$$
\Phi_i=\hat\Phi_i+V_i=\begin{pmatrix}\phi_i^+\\ v_i+\rho_i+i\psi_i\end{pmatrix}\ , \tag{6.62a}
$$

$$
\Phi_i^*=\hat\Phi_i^*+V_i=\begin{pmatrix}\phi_i^-\\ v_i+\rho_i-i\psi_i\end{pmatrix}\ . \tag{6.62b}
$$

We note that the terms linear in fluctuations are a consequence of the non-Hermitian nature of the system. However as we discussed before, they do not play a role in the equations of motion $\delta S/\delta\hat\Phi_i^\dagger\equiv 0$, since these therms depend on $\hat\Phi_i$ only. The equations of motion are given by

$$
\begin{aligned}
0 =& \Box\hat{\Phi}_1 - \frac{\mu^4}{m_2^2}\hat{\Phi}_1 + \frac{\kappa}{2}\left(V_1^\dagger\hat{\Phi}_1 + \hat{\Phi}_1^\dagger V_1\right)V_1 + \mu^2\hat{\Phi}_2 && (6.63a)\\
&+ \frac{\kappa}{2}|\hat{\Phi}_1|^2 V_1 + \frac{\kappa}{2}\left(V_1^\dagger\hat{\Phi}_1 + \hat{\Phi}_1^\dagger V_1\right)\hat{\Phi}_1 + \frac{\kappa}{2}|\hat{\Phi}_1|^2\hat{\Phi}_1 \ ,\\
0 =& \Box\hat{\Phi}_2 + m_2^2\hat{\Phi}_2 - \mu^2\hat{\Phi}_1 \ . && (6.63b)
\end{aligned}
$$

The massless Goldstone modes consist of charged and neutral fields:

$$
\begin{aligned}
G^\pm =& \frac{1}{\sqrt{v_1^2 - v_2^2}}\left(v_1\phi_1^\pm - v_2\phi_2^\pm\right) \ , && (6.64a)\\
G =& \frac{1}{\sqrt{v_1^2 - v_2^2}}\left(v_1\psi_1 - v_2\psi_2\right) \ . && (6.64b)
\end{aligned}
$$

The remaining fields consist of a charged field and three neutral fields. The charged fields are given by

$$
H^\pm = \frac{1}{\sqrt{v_1^2 - v_2^2}}\left(v_2\phi_1^\pm - v_1\phi_2^\pm\right) \ , \qquad (6.65)
$$

and one neutral field is given by

$$
D = \frac{1}{\sqrt{v_1^2 - v_2^2}}\left(v_2\psi_1 - v_1\psi_2\right) \ , \qquad (6.66)
$$

with degenerate squared mass

$$
M^2 = \frac{v_1^2 - v_2^2}{v_1 v_2}\mu^2 = m_2^2 - \frac{\mu^4}{m_2^2} \ . \qquad (6.67)
$$

Finally, we can express the last two neutral fields as

$$
\begin{aligned}
H =& \rho_1\cosh\alpha - \rho_2\sinh\alpha \ , && (6.68a)\\
h =& \rho_1\sinh\alpha - \rho_2\cosh\alpha \ , && (6.68b)
\end{aligned}
$$

with masses

$$\begin{aligned} M_h^2 =& \frac{1}{2}\left(m_2^2+2m_1^2-3\mu^4/m_2^2-\sqrt{\left(2m_1^2-m_2^2-3\mu^4/m_2^2\right)^2-4\mu^4}\right) &\text{(6.69a)}\\ =& \left(v_1^2-v_2^2\right)\left[\lambda-\frac{\hat{\lambda}\cosh\left(\beta-\alpha\right)}{\sinh\left(\beta-\alpha\right)}\right] ,\\ M_H^2 =& \frac{1}{2}\left(m_2^2+2m_1^2-3\mu^4/m_2^2+\sqrt{\left(2m_1^2-m_2^2-3\mu^4/m_2^2\right)^2-4\mu^4}\right) &\text{(6.69b)}\\ =& \left(v_1^2-v_2^2\right)\left[\lambda-\frac{\hat{\lambda}\sinh\left(\beta-\alpha\right)}{\cosh\left(\beta-\alpha\right)}\right] , \end{aligned}$$

where

$$\tanh\alpha = \frac{-\mu^2}{\left(M_H^2-m_2^2\right)} , \tag{6.70a}$$

$$\tanh\beta = \frac{v_2}{v_1} , \tag{6.70b}$$

and

$$\lambda = \kappa\cosh^4\beta , \tag{6.71a}$$

$$\hat{\lambda} = \frac{\kappa}{2}\sinh 2\beta\cosh^2\beta . \tag{6.71b}$$

It is not obvious that M^2 is positive or that M_H^2 and M_h^2 are real, and we derive the corresponding conditions on μ^2 for this to be the case in the next Section.

For this $\mathscr{P}\mathscr{T}$-symmetric theory, the eigenmodes of this non-Hermitian Hamiltonian are orthogonal with respect to the $\mathscr{C}'\mathscr{P}\mathscr{T}$ inner product

$$\langle\phi,\varphi\rangle_{\mathscr{C}'\mathscr{P}\mathscr{T}} = \int_x\left(\phi^{\mathscr{C}'\mathscr{P}\mathscr{T}}\right)^{\mathsf{T}}\varphi , \tag{6.72}$$

and the fields $G^\pm$, G, $H^\pm$, D, H and h are normalized accordingly. These eigenmodes are non-trivial linear combinations of the scalar components of Φ_1 and the pseudoscalar components of Φ_2 and, as such, they cannot be eigenstates of $\mathscr{P}$.

We remark that the $\mathscr{C}'\mathscr{P}\mathscr{T}$ norm used for the modes G, $G^\pm$, D and $H^\pm$ in equation (6.64), (6.65) and (6.66) diverges when $\mu^2 = \pm m_2^2$ ($v_1^2 = v_2^2$). At this point — the *zero exceptional point* described in [80] — we lose three eigendirections: $D \propto G$ and $H^\pm \propto G^\pm$. On the other hand, when $\mu^2 = \pm T_{H(h)}$, where

$$T_{H(h)} = \frac{m_2^2}{9}\left(6m_1^2 - m_2^2 + (-)2\sqrt{2m_2^2\left(3m_1^2 - m_2^2\right)}\right), \tag{6.73}$$

$|\alpha| \to \infty$ and the $\mathscr{PT}$ norm of h and H in Eq. (6.68) diverges. In this case, we lose one eigendirection: $H \propto h$. We discuss this exceptional point further in Subsection 6.2.3.4.

6.2.3.2 Conditions on μ^2

Ensuring that we are in a physical regime of spontaneous symmetry breaking leads to a number of constraints on the parameter μ^2:

I In order for the symmetry to be broken [see equation (6.54)], we require that

$$\mu^4 < m_1^2 m_2^2 . \tag{6.74}$$

II In order to ensure that the squared mass M^2, defined in equation (6.67), remains positive, we require that

$$\mu^4 < m_2^4 . \tag{6.75}$$

III In order for the squared masses M_h^2 and M_H^2, defined in equation (6.69), to be real, we require that

$$4\mu^4 \leq \left(2m_1^2 - m_2^2 - \frac{3\mu^4}{m_2^2}\right)^2 . \tag{6.76}$$

We remark that in the region $4\mu^4 \geq \left(2m_1^2 - m_2^2 - \frac{3\mu^4}{m_2^2}\right)^2$ the squared mass matrix cannot be brought to a Hermitian form by a similarity transformation [58].

These constraints on the parameter μ^4 are plotted in figure 6.1. The unshaded regions correspond to values of μ^4 consistent with a physical spontaneous symmetry-breaking phase, satisfying all of the previously mentioned conditions. The various constraints on μ^4 can be summarised as follows:

- If $m_2^2 < \frac{m_1^2}{3}$ then $\mu^4 < m_2^4$ (Condition II);
- If $\frac{m_1^2}{3} < m_2^2 < m_1^2$ then $\mu^4 < T_h$ (Condition III);
- If $m_1^2 < m_2^2 < 3m_1^2$ then $\mu^4 < T_h$ (Condition III) or $T_H < \mu^4 < m_1^2 m_2^2$ (Conditions I and III);
- If $3m_1^2 < m_2^2$ then $\mu^4 < m_1^2 m_2^2$ (Condition I).

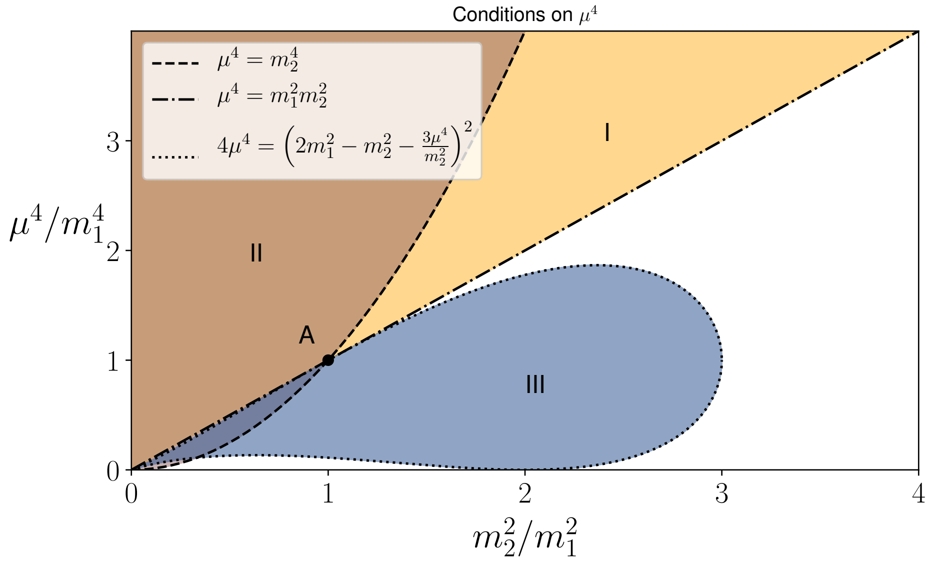

Figure 6.1: The excluded regions for the parameter μ^4, corresponding to the constraints I, II and III, plotted as functions of m_2^2/m_1^2. Region I corresponds to the symmetric phase of the $U(1) \times SU(1)$ symmetry [see equation (6.74)]. Region II corresponds to the broken phase of $\mathscr{PT}$ symmetry [see equation (6.75)] in which M^2 is negative. Region III corresponds to the broken phase of $\mathscr{PT}$ symmetry in which M_h^2 and M_H^2 are complex [see equation (6.76)]. The unshaded region corresponds to a physical SSB phase for the $U(1) \times SU(2)$ symmetry. For $m_2^2/m_1^2 < 1/3$, the allowed region is determined only by condition II. For $m_1^2/3 < m_2^2 < 3m_1^2$, the allowed region is determined by conditions I and III. Lastly, in the region $m_2^2 > 3m_1^2$, the allowed region is determined only by condition III. At the point A, all the conditions become equivalent.

6.2.3.3 Equations of Motion after SSB

After expressing the full Lagrangian in terms of fluctuations around the vevs, as was done in equation (6.62), we can now express the equations of motion after symmetry breaking in terms of the gauge fields Z^α, W^α and A^α. Introducing the notations

$$C_+^\alpha \equiv \frac{J_{+,1}^\alpha - iJ_{+,2}^\alpha}{\sqrt{2}} , \qquad K_+^\alpha \equiv J_{+,3}^\alpha \cos\theta_\mathrm{W} - I_+^\alpha \sin\theta_\mathrm{W} , \tag{6.77}$$

and

$$\begin{aligned} \sigma &\equiv \mathbb{I} + \tau_3 \quad , & \omega &\equiv \frac{\tau_3 \cos^2\theta_\mathrm{W} - \sin^2\theta_\mathrm{W}}{\cos\theta_\mathrm{W}} , \\ \tau_+ &\equiv \frac{\tau_1 - i\tau_2}{\sqrt{2}} \quad , & \tau_- &\equiv \frac{\tau_1 + i\tau_2}{\sqrt{2}} , \end{aligned} \tag{6.78}$$

the equations of motion read as follows:

Scalar fields

$$\begin{aligned} 0 = D_\alpha D^\alpha \hat{\Phi}_1 + D_\alpha \left(\frac{ig}{2} Z^\alpha \omega + \frac{ig}{2} W^\alpha \tau_- \right) V_1 - \frac{\mu^4}{m_2^2}\hat{\Phi}_1 \qquad\qquad (6.79\text{a}) \\ + \frac{\kappa}{2}\left(V_1^\dagger \hat{\Phi}_1 + \hat{\Phi}_1^\dagger V_1\right) V_1 + \mu^2 \hat{\Phi}_2 + \frac{\kappa}{2}|\hat{\Phi}_1|^2 V_1 + \frac{\kappa}{2}\left(V_1^\dagger \hat{\Phi}_1 + \hat{\Phi}_1^\dagger V_1\right)\hat{\Phi}_1 + \frac{\kappa}{2}|\hat{\Phi}_1|^2 \hat{\Phi}_1 , \end{aligned}$$

$$0 = D_\alpha D^\alpha \hat{\Phi}_2 + D_\alpha \left(\frac{ig}{2} Z^\alpha \omega + \frac{ig}{2} W^\alpha \tau_- \right) V_2 + m_2^2 \hat{\Phi}_2 - \mu^2 \hat{\Phi}_1 ; \tag{6.79b}$$

Z^α gauge field

$$\begin{aligned} 0 =& \partial_\beta Z^{\alpha\beta} + ig\cos\theta_\mathrm{W}\left(W_\beta^\dagger W^{\beta\alpha} - W^{\dagger\beta\alpha}W_\beta\right) + \frac{1}{\xi}\partial^\alpha\partial^\beta Z_\beta \\ &+ \frac{g^2}{2\cos^2\theta_\mathrm{W}}\left(|V_1|^2 + |V_2|^2\right) Z^\alpha - K_+^\alpha + ig\cos\theta_\mathrm{W}\left(\partial^\alpha \bar{\chi}^\dagger \chi - \partial^\alpha \bar{\chi}\chi^\dagger\right) \\ &+ \frac{g^2}{2} Z^\alpha \sum_i \left(\hat{\Phi}_i^\dagger \omega^2 \hat{\Phi}_i + \left[V_i^\dagger \hat{\Phi}_i + \hat{\Phi}_i^\dagger V_i\right]\right) + \frac{eg}{2} A^\alpha \sum_i \Phi_i^\dagger (\omega\sigma) \hat{\Phi}_i \\ &- \frac{gg'}{2}\sin\theta_\mathrm{W} \sum_i \left(\left[\hat{\Phi}_i^\dagger \tau_- V_i\right] W^\alpha + \left[V_i^\dagger \tau_+ \hat{\Phi}_i\right] W^{\alpha\dagger}\right) \\ &- \frac{gg'}{2}\sin\theta_\mathrm{W} \sum_i \left(\left[\hat{\Phi}_i^\dagger \tau_- \hat{\Phi}_i\right] W^\alpha + \left[\hat{\Phi}_i^\dagger \tau_+ \hat{\Phi}_i\right] W^{\alpha\dagger}\right) ; \end{aligned} \tag{6.80}$$

A^α gauge field

$$\begin{aligned}0 =&\partial_\beta F^{\alpha\beta} + ig\sin\theta_\mathrm{W}\left(W^\dagger_\beta W^{\beta\alpha} - W^{\dagger\beta\alpha}W_\beta\right) + \frac{1}{\xi}\partial^\alpha\partial^\beta A_\beta \\ &- Q^\alpha + ig\sin\theta_\mathrm{W}\left(\partial^\alpha\bar{\chi}^\dagger\chi - \partial^\alpha\bar{\chi}\chi^\dagger\right) \\ &+ e^2A^\alpha\sum_i\hat{\Phi}^\dagger_i\sigma\,\hat{\Phi}_i + \frac{eg}{2}Z^\alpha\sum_i\hat{\Phi}^\dagger_i\,(\omega\sigma)\,\hat{\Phi}_i \\ &+ \frac{eg}{2}\sum_i\left(\left[\hat{\Phi}^\dagger_i\tau_-\hat{\Phi}_i\right]W^\alpha + \left[\hat{\Phi}^\dagger_i\tau_+\hat{\Phi}_i\right]W^{\alpha\dagger} + \left[\hat{\Phi}^\dagger_i\tau_- V_i\right]W^\alpha + \left[V^\dagger_i\tau_+\hat{\Phi}_i\right]W^{\alpha\dagger}\right)\ ;\end{aligned} \tag{6.81}$$

W^α gauge fields

$$\begin{aligned}0 =&\partial_\beta W^{\alpha\beta} + igW_\beta\left(\sin\theta_\mathrm{W}F^{\beta\alpha} + \cos\theta_\mathrm{W}Z^{\beta\alpha}\right) + \frac{1}{\xi}\partial^\alpha\partial^\beta W_\beta \\ &- ig\left(\sin\theta_\mathrm{W}A_\beta + \cos\theta_\mathrm{W}Z_\beta\right)W^{\beta\alpha} \\ &+ \frac{g^2}{2}W^\alpha\left(|V_1|^2 + |V_2|^2\right) - C^\alpha_+ + ig\left(\partial^\alpha\bar{\chi}\eta_3 - \partial^\alpha\bar{\eta}_3\chi\right) \\ &+ \frac{g^2}{2}W^\alpha\sum_i\left(V_i\hat{\Phi}_i + \hat{\Phi}^\dagger_i V_i + |\hat{\Phi}_i|^2\right) \\ &+ \frac{g}{2}\left(eA^\alpha - g'\sin\theta_\mathrm{W}Z^\alpha\right)\sum_i\left(\hat{\Phi}_i\tau_+\hat{\Phi}_i + V^\dagger_i\tau_+\hat{\Phi}_i\right)\ .\end{aligned} \tag{6.82}$$

From these equations, we can see that the gauge field masses are

$$M_W = g\sqrt{\frac{v_1^2+v_2^2}{2}} = \cos\theta_\mathrm{W}M_Z \qquad \text{and} \qquad M_A = 0\ , \tag{6.83}$$

as in the Hermitian Standard Model.

6.2.3.4 Comments on the Exceptional Points

At the zero exceptional points $\mu^2 = \pm m_2^2$, the vevs become

$$v_1^2 = v_2^2 \equiv v^2 = \frac{2}{\kappa}(m_1^2 - m_2^2)\ , \tag{6.84}$$

which vanish in the degenerate limit $m_1^2 = m_2^2$. For $m_1^2 \neq m_2^2$, though, the gauge boson masses at the exceptional points are

$$M_W^2 = g^2 v^2 = \cos^2\theta_{\mathrm{W}} M_Z^2 \neq 0 \,, \tag{6.85}$$

remaining physical and non-zero.

In order to make sense of this, in spite of the divergence of the $\mathscr{C}'\mathscr{P}\mathscr{T}$ norm and the apparent non-normalisability of the Goldstone modes (see equation (6.64)), it is helpful to reconsider the behaviour of the non-Hermitian theory at the exceptional point. As an example, let us consider the 2×2 squared mass matrix of the non-interacting theory we discussed in previous sections

$$\mathbf{M}^2 = \begin{pmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2 \end{pmatrix} . \tag{6.86}$$

For $m_1^2 > m_2^2$, the eigenvectors of this squared mass matrix are

$$\mathbf{e}_+ = N \begin{pmatrix} \eta \\ \sqrt{1-\eta^2}-1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{e}_- = N \begin{pmatrix} 1-\sqrt{1-\eta^2} \\ -\eta \end{pmatrix} , \tag{6.87}$$

where

$$\eta = \frac{2\mu^2}{m_1^2 - m_2^2} \tag{6.88}$$

(not to be confused with the ghost field appearing earlier). The eigenvectors are not orthogonal with respect to the usual Hermitian inner product:

$$\mathbf{e}_+^* \cdot \mathbf{e}_- = 2N^2\eta(1-\sqrt{1-\eta^2}) \,, \tag{6.89}$$

except in the Hermitian limit $\mu \to 0$ ($\eta \to 0$). They are, however, orthogonal with respect to the $\mathscr{P}\mathscr{T}$ inner product, and orthonormality fixes

$$N = \left(2\eta^2 - 2 + 2\sqrt{1-\eta^2}\right)^{-1/2} . \tag{6.90}$$

The exceptional point of this mass matrix occurs when $\eta \to 1$, at which point the normalisation of the eigenvectors diverges (see figure 3.1). This signals that the mass matrix has become defective, having the Jordan normal form

$$M^2\big|_{\eta\to 1} = \begin{pmatrix} (m_1^2+m_2^2)/2 & 1 \\ 0 & (m_1^2+m_2^2)/2 \end{pmatrix}, \tag{6.91}$$

and we lose an eigenvector. In fact, we see that in the limit $\eta \to 1$ the eigenvectors $\mathbf{e}_+$ and $\mathbf{e}_-$ become parallel to one another (see also figure 3.1) However, the issue of the non-orthogonality of these eigenvectors is then moot, and we can normalise them with respect to the Hermitian inner product, fixing

$$N\big|_{\eta=1} = \frac{1}{\sqrt{2}}\,. \tag{6.92}$$

In other words, at the exceptional point, the system behaves like a Hermitian theory with one fewer degree of freedom.

Returning to the case of spontaneously-broken gauge symmetries at the zero exceptional points, the explanation for the non-vanishing masses of the gauge bosons is that the Goldstone modes must be normalised with respect to Hermitian conjugation and not $\mathscr{PT}$ conjugation (which has become ill-defined). The discontinuity in the behaviour of the system as we approach exceptional points means that we must treat these particular points separately in parameter space.

Thus, our conclusion is that it is possible to give masses to gauge bosons in a gauge-invariant way through SSB also for non-Hermitian theories, even at the exceptional points. At these points, however, the counting of eigendirections must allow for the fact that the Hamiltonian has become defective.

We note that different results were derived in [58, 90, 91], which is based on an alternative interpretation of a similar version of this non-Hermitian theory, and where the gauge boson masses are zero at the zero exceptional point. The difference in our results can be traced back to differing interpretations of the complex conjugate: we take complex conjugation to act linearly on the fields, whereas in [58] it is taken to act antilinearly on one of the fields (as motivated by a similarity transformation to a Hermitian theory). This has the effect of interchanging $v_2^2 \to -v_2^2$ in the expression for the gauge boson masses, such that they then vanish at the zero exceptional point when $v_1^2 = v_2^2$. It is then argued that this is consistent with the fact that the Goldstone modes cannot be normalised with respect to the $\mathscr{PT}$ norm, which diverges at exceptional points, and these modes, therefore, cannot be "eaten" by the gauge field. This then leads [58] to conclude that it is possible to break the gauge symmetry of a non-Hermitian model spontaneously without giving a mass to the gauge bosons. Our conclusion is the opposite: the gauge boson remains massive in the symmetry-broken phase, even at the zero exceptional point. Remark that this mirrors the difference in the masses for the $U(1)$ gauge fields as discussed in Chapter 4.3.2

6.2.4 Masses in the Non-Hermitian Model Compared with the Hermitian Model

We discuss the dependencies of the scalar and vector masses in the non-Hermitian 2HDM on the non-Hermitian mixing parameter μ^2. These dependencies are shown in figures 6.2 and 6.3 for the scalar and vector bosons, respectively, wherein we have introduced the notation $\beta_{H(h)} \equiv T_{H(h)}/m_2^4$.

Additionally, we compare these masses to those of a similar system like the one discussed in Section 6.1. For this model, we plot the dependence of the scalar and vector masses on the Hermitian mixing parameter m_{12}^2.

We note the following features from each panel of figure 6.2:

- In the region $m_1^2 > 3m_2^2$, the mass M^2 goes to zero at the exceptional point $\mu^2 = m_2^2$. If μ^2 were to become larger then m_2^2 then M^2 would become negative, and we would enter the phase of broken $\mathscr{PT}$ symmetry.

- In the region $m_1^2/3 < m_2^2 < m_1^2$, the masses M_H^2 and M_h^2 become equal at the point $\tanh^2\beta = \beta_h$. For larger values of μ^2, both M_H^2 and M_h^2 would become complex.

- For $m_1^2 < m_2^2 < 3m_1^2$, the masses M_H^2 and M_h^2 become equal at the point $\tanh^2\beta = \beta_h$ or $\tanh^2\beta = \beta_H$. Between these points, M_H^2 and M_h^2 become complex. When $\tanh^2\beta > m_1^2/m_2^2$, the mass M_H^2 becomes negative. The unshaded regions correspond to physical masses.

- For $m_2^2 > 3m_1^2$, the masses are all real and positive as long as $\tanh^2\beta < m_1^2/m_2^2$.

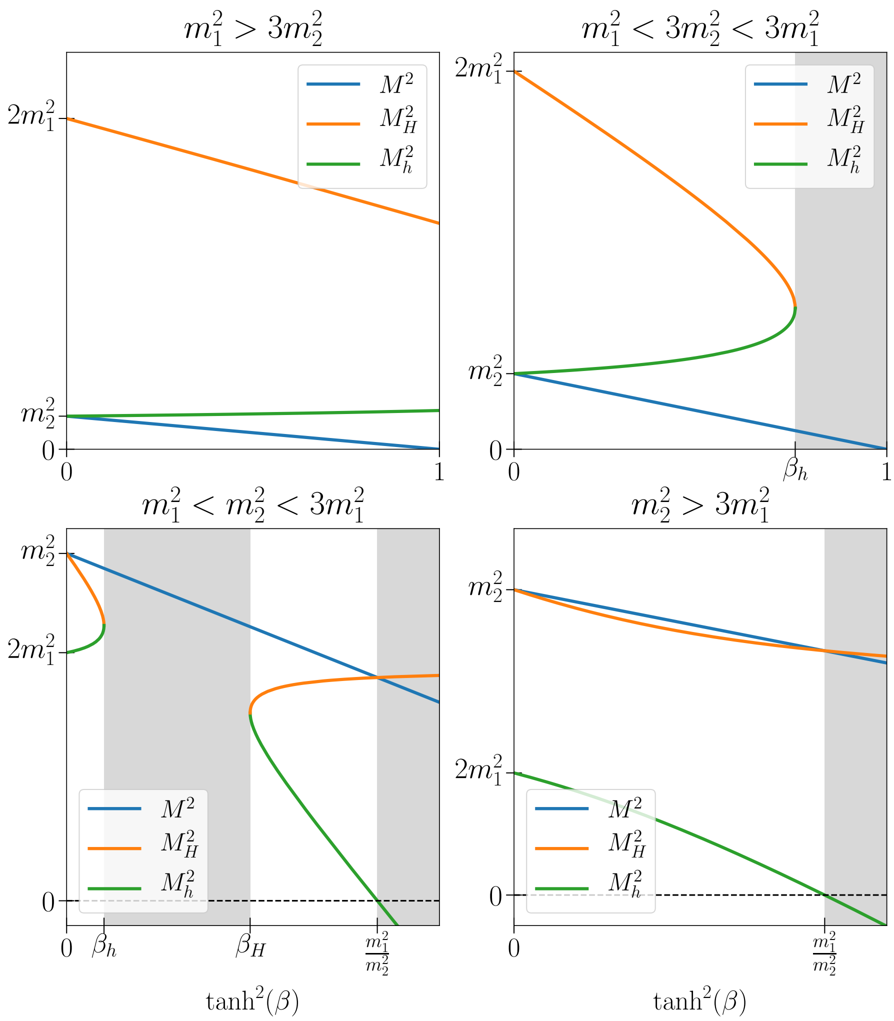

Figure 6.2: The masses of the physical scalar bosons as functions of $\tanh^2\beta$ in different parameter regions. Unphysical parameter regions are shaded grey. The upper left panel shows the region where $m_1^2 > 3m_2^2$, the upper right panel shows the region where $m_1^2 < 3m_2^2 < 3m_1^2$, the lower left panel shows the region where $m_1^2 < m_2^2 < 3m_1^2$, and the lower right panel shows the region where $m_2^2 > 3m_1^2$.

We note in the lower right panel of figure 6.3 that the gauge-boson masses vanish at the point $\mu^4 = m_1^2 m_2^2$, where the symmetry is restored, as we would expect.

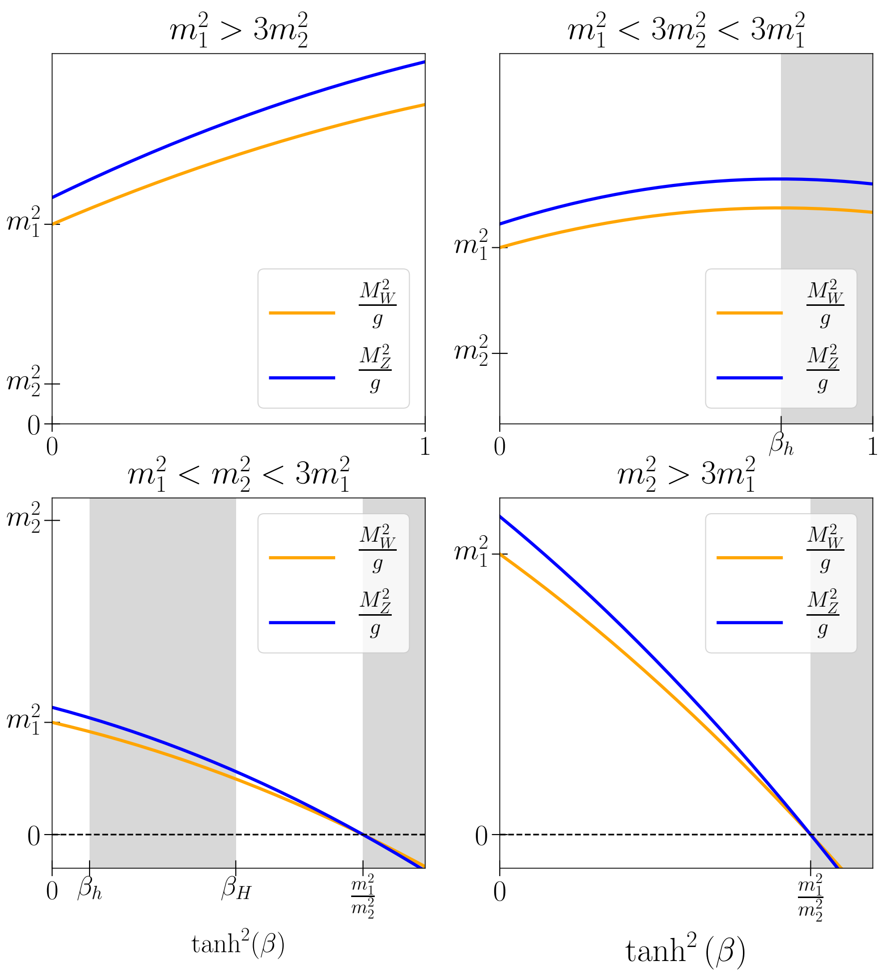

Figure 6.3: The masses of the charged and neutral gauge bosons as functions of $\tanh^2\beta$ in the same parameter regions as in Fig. 6.2. Unphysical parameter regions are shaded grey.

The masses of the $\mathscr{PT}$-symmetric non-Hermitian model can be compared to those of the Hermitian 2HDM Lagrangian we discussed in section 6.1 after we take the limit $\kappa_2, \kappa_3, \kappa_4, \kappa_5 \to 0 \ \ , \ \ \kappa_1 \to \kappa$:

$$\begin{aligned} \mathscr{L} \ &= \ \partial_\alpha \Phi_1^\dagger \partial^\alpha \Phi_1 + \partial_\alpha \Phi_2^\dagger \partial^\alpha \Phi_2 + m_1^2 |\Phi_1|^2 - m_2^2 |\Phi_2|^2 \\ &\quad + m_{12}^2 \left(\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1 \right) - \frac{\kappa}{4} |\Phi_1|^4 \, . \end{aligned} \tag{6.93}$$

The vacuum expectation values for this Lagrangian are

$$\langle \Phi_1 \rangle = \begin{pmatrix} 0 \\ v_1^H \end{pmatrix} = V_1^H \,, \qquad \langle \Phi_2 \rangle = \begin{pmatrix} 0 \\ v_2^H \end{pmatrix} = V_2^H \,, \tag{6.94}$$

with

$$v_1^H = \sqrt{\frac{2}{\kappa}\left(m_1^2 + \frac{m_{12}^4}{m_2^2}\right)} \,, \qquad v_2^H = \frac{m_{12}^2}{m_2^2}\sqrt{\frac{2}{\kappa}\left(m_1^2 + \frac{m_{12}^4}{m_2^2}\right)} \,. \tag{6.95}$$

After expressing the Lagrangian in terms of the shifted field $\hat{\Phi}_i$ where

$$\Phi_i = \hat{\Phi}_i + V_i^H = \begin{pmatrix} \phi_i^+ \\ v_i^H + \rho_i + i\psi_i \end{pmatrix} \,, \tag{6.96a}$$

$$\Phi_i^* = \hat{\Phi}_i^* + V_i^H = \begin{pmatrix} \phi_i^- \\ v_i^H + \rho_i - i\psi_i \end{pmatrix} \,, \tag{6.96b}$$

we can calculate the eigenvalues. As in the non-Hermitian model, the massless states consist of massless charged scalar and pseudoscalar Goldstone fields

$$G^\pm = \frac{1}{\sqrt{(v_1^H)^2 + (v_2^H)^2}} \left(v_1^H \phi_1^\pm + v_2^H \phi_2^\pm \right) \,, \tag{6.97a}$$

$$G = \frac{1}{\sqrt{(v_1^H)^2 + (v_2^H)^2}} \left(v_1^H \psi_1 + v_2^H \psi_2 \right) \,. \tag{6.97b}$$

The normalisations of the eigenmodes should be compared with those in equation (6.64). We remark that this Hermitian model is not $\mathscr{P}\mathscr{T}$ symmetric if Φ_1 and Φ_2 transform as a scalar and a pseudoscalar, respectively. It is, however, $\mathscr{P}\mathscr{T}$ symmetric if both Φ_1 and Φ_2 transform as scalars or pseudoscalars, and the Hermitian and $\mathscr{P}\mathscr{T}$ norms coincide, as is expected for a Hermitian, $\mathscr{P}\mathscr{T}$-symmetric theory.

The remaining massive fields include a charged scalar, a neutral pseudoscalar and two neutral scalar fields. The charged scalars are

$$H^{\pm} = \frac{-1}{\sqrt{(v_1^H)^2 + (v_2^H)^2}} \left(v_2^H \phi_1^{\pm} - v_1^H \phi_2^{\pm} \right) , \tag{6.98}$$

and the pseudoscalar is

$$D = \frac{-1}{\sqrt{(v_1^H)^2 + (v_2^H)^2}} \left(v_2^H \psi_1 - v_1^H \psi_2 \right) , \tag{6.99}$$

with degenerate squared mass

$$M^2 = \frac{(v_1^H)^2 + (v_2^H)^2}{v_1^H v_2^H} m_{12}^2 . \tag{6.100}$$

Lastly, we can express the neutral scalar boson fields as

$$\begin{aligned} H &= -\rho_1 \cos\alpha - \rho_2 \sin\alpha , & (6.101a) \\ h &= \rho_1 \sin\alpha - \rho_2 \cos\alpha , & (6.101b) \end{aligned}$$

with squared masses

$$\begin{aligned} M_h^2 &= \left((v_1^H)^2 + (v_2^H)^2\right) \left[\lambda - \frac{\hat{\lambda} \cos(\beta - \alpha)}{\sin(\beta - \alpha)} \right] , & (6.102a) \\ M_H^2 &= \left((v_1^H)^2 + (v_2^H)^2\right) \left[\lambda + \frac{\hat{\lambda} \sin(\beta - \alpha)}{\cos(\beta - \alpha)} \right] , & (6.102b) \end{aligned}$$

where

$$\begin{aligned} \tan\alpha &= \frac{-m_{12}^2}{\left(M_H^2 - m_2^2\right)} , & (6.103a) \\ \tan\beta &= \frac{v_2^H}{v_1^H} , & (6.103b) \end{aligned}$$

and

$$\lambda = \kappa \cos^4\beta \,, \tag{6.104a}$$
$$\hat{\lambda} = \frac{\kappa}{2}\sin 2\beta \cos^2\beta \,. \tag{6.104b}$$

The squared masses for this Hermitian model are plotted in figure 6.4 in the parameter ranges $2m_1^2 > m_2^2$ (left panel) and $2m_1^2 < m_2^2$ (right panel). We see that the mass spectra are completely different from the non-Hermitian, $\mathscr{P}\mathscr{T}$-symmetric case, offering distinctive phenomenological possibilities.

Before concluding, we remark that, by comparing the expressions above with those in Subsection 6.2.3.1, we can see that the non-Hermitian 2HDM that we have considered in this work is an analytic continuation of the Hermitian 2HDM, obtained by taking $m_{12}^4 \to -\mu^4$. In other words, the Hermitian 2HDM lies in the fourth quadrant of the $(m_2^2/m_1^2, \mu^4/m_1^4)$ plane, not shown in figure 6.1.

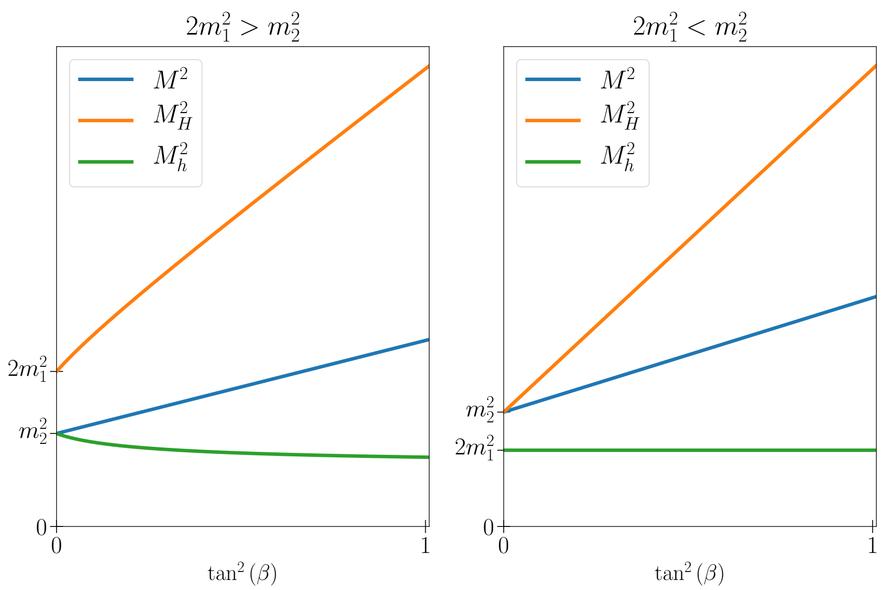

Figure 6.4: The masses of the scalar fields in the Hermitian two-Higgs-doublet model as functions of $\tan^2\beta$ in the parameter ranges $2m_1^2 > m_2^2$ (left panel) and $2m_1^2 < m_2^2$ (right panel).

6.3 Conclusion

In this section, we discussed non-Abelian spontaneous symmetry breaking for our non-Hermitian model. For the field of our system to transform under the non-Abelian $SU(2)$ transformation, we will upgrade the complex scalar fields ϕ_i $(i = 1,2)$ to complex scalar doublets Φ_i.

We start this section by discussing a Hermitian two-Higgs-doublet model with a doublet mixing term of the form $m_{12}^2\left(\Phi_1^\dagger\Phi_2 + \Phi_2^\dagger\Phi_1\right)$. We calculated the mass spectrum for this system explicitly and their corresponding eigenvectors. We showed that this model possesses a charged and a neutral Goldstone boson and two charged and three neutral Higgs bosons.

Lastly, we discuss a non-Hermitian two-Higgs-doublet model. To consistently gauge this Lagrangian, we need to introduce covariant derivatives. This is done in such a way that couples the gauge fields to non-conserved currents. Because these are not conserved, gauge fixing terms need to be introduced in the Lagrangian. We derived the gauge restrictions explicitly by utilising the BRST symmetry of the gauge fixed system. After having obtained a consistent $U(1)\times SU(2)$ gauged theory, we calculated the spectrum after spontaneous symmetry breaking. Afterwards, we are able to derive the physical region in parameter space. We are then able to examine our system at its critical limits of this region. We find for our model, that at this critical limit the Englert-Brout-Higgs mechanism still holds and the gauge fields remain massive. Lastly, we compare the masses of this non-Hermitian 2HDM with a Hermitian 2HDM. We see how the mass spectrums of these two theories differ from each other. From this spectrum, it follows that we can see our non-Hermitian model as an analytic continuation of the Hermitian one.

Chapter 7

General Conclusions

In this thesis, we discussed how to consistently construct a non-Hermitian Quantum Field Theory. We propose a non-Hermitian extension to the Higgs sector in our Standard Model. The construction of this model is especially non-trivial for the equations of motion, formulation of conserved currents and gauge invariance. We formulate consistent methods to deal with these problems.
We will here summarise our main finding and briefly discuss possible further works.

In Chapter 3, we introduced a non-Hermitian scalar field Lagrangian consisting of two complex scalar fields. We have shown that this system is $\mathscr{PT}$-symmetric, and the spectrum is real within a particular region. Inside this region, our system possesses unbroken-$\mathscr{PT}$ symmetry. Because of the non-Hermitian nature of this system, minimising the action to find consistent equations of motion is not trivial. Where the classical equations of motion typically are found by varying the action with respect to the field and their complex conjugate, we now need to choose one of these two options. Note that this approach differs from the one that was used in works such as [58, 80, 90, 91] for similar systems. These different approaches do lead to different physical results, as we saw in Chapter 4 and Chapter 6.

For our approach, a different choice in equations of motion will not lead to different physical observables. This will, however, have repercussions when we reexamine the connection between conserved currents and symmetries. Because these different choices in equations of motion cannot both simultaneously be satisfied, the normal Noether's current of a symmetry is no longer conserved. Instead, the conserved current corresponds to a transformation that changes the Lagrangian is a specific way. Afterwards, we expressed the Lagrangian after spontaneous symmetry breaking and show that this system possesses a Nambu-Goldstone mode. So we find that the Goldstone theorem still holds for our non-Hermitian system.

Then we also discuss local transformations and symmetries of this system in Chapter 4. It turns out that in order to have a consistent polarisation tensor, the gauge field should couple to a non-conserved current. This leads however to inconsistent Maxwell equations. This problem can be resolved if we also introduce gauge fixing terms into our Lagrangian. These gauge fixing terms must be added at the classical level, and thus our system only possesses restricted gauge freedom. Once the model has been successfully gauged, we have shown that the Englert-Brout-Higgs for mass generalisation of the gauge boson still holds.

We moved on into Chapter 5 and discussed how the path integral should be constructed. It turns out that when doing this, we need to make sure that we integrate over $\mathscr{C}'\mathscr{P}\mathscr{T}$ conjugate field variables and include $\mathscr{C}'\mathscr{P}\mathscr{T}$ conjugate source fields. We use this formalism to show the running of the couplings for our scalar model using the one-loop $1PI$ effective action. Finally, we also calculate the Goldstone mode at the one-loop order and check the reality of gauge field after taking quantum corrections into account.

In the final Chapter 6 we want to build upon the model introduced in Chapter 4 by also including $SU(2)$ gauge fields. This way, we naturally end up with a non-Hermitian two-Higgs-doublet model. We are then able to follow a similar method as in Chapter 4 to gauge this model. We derived the conserved currents for this gauged model. Similar as was the case for the Abelian gauge field, the non-Abelian gauge fields must couple to non-conserved currents, and we need to introduce non-Abelian gauge fixing terms into our Lagrangian. These non-Abelian gauge restrictions are then explicitly derived using the BRST symmetry of our gauge fixed Lagrangian. Once we constructed this consistent model, we calculate the spectrum after spontaneous symmetry breaking. Afterwards, we are able to find the physical range in parameter space that allows for real energies. The critical points in this space are of particular interest. At this critical point, the different eigenvectors become parallel, and our model reverts to a Hermitian one. We show that in this limit the gauge fields do still possess a non-zero mass. After having derived the eigenmasses of this non-Hermitian two-Higgs-doublet model, we then compared these to the eigenmasses of a Hermitian two-Higgs-doublet model. These two models posses different spectrums, and we can see that the non-Hermitian model is an analytic continuation of the Hermitian model. An interesting difference between our approach and the one followed in [58, 80, 90, 91], is that in these works the Englert-Brout-Higgs fails at this critical point.

Remark that the non-Hermitian two-Higgs-doublet model we introduced has a significant different mass spectrum compared to the Hermitian version. This opens up new phenomenological perspectives that would prove interesting to study further. In this work we laid the groundwork for Quantum Field Theory model building. The model in this work naturlly introduces a $\mathscr{P}$-odd pseudo-scalar. This might lead itself to an interesting extionsion to the non-Abelian Higgs-Axion model.
It would also be interesting to expand this model by also including fermionic fields and Yukawa interactions. Furthermore one can look into non-Hermitian extensions of the Yukawa sector [49, 50, 92]. In the work [93] a non-Hermitian Quantum Field Theory with supersymmetric theory was discussed.

Appendix A

Appendices

A.1 Running Couplings

The full bare potential is

$$\begin{aligned} U^{(0)} &= m_1^2|\phi_1|^2 + m_2^2|\phi_2|^2 + \mu^2(\phi_1^\star\phi_2 - \phi_2^\star\phi_1) \\ &\quad + \frac{g_1}{4}|\phi_1|^4 + \frac{g_2}{4}|\phi_2|^4 + \lambda|\phi_1\phi_2|^2 + \frac{\alpha}{4}\left((\phi_1^\star\phi_2)^2 + (\phi_2^\star\phi_1)^2\right) \\ &\quad + \frac{1}{2}\left(\beta_1|\phi_1|^2 + \beta_2|\phi_2|^2\right)\left(\phi_1^\star\phi_2 - \phi_2^\star\phi_1\right) , \end{aligned} \tag{A.1}$$

and the one-loop 1PI potential is given by

$$U^{(1)} = U^{(0)} + \frac{1}{2V^{(4)}}\,\mathrm{STr}\ln S_E^{(2)} , \tag{A.2}$$

where

$$S_E^{(2)} = \begin{pmatrix} p^2 + U^{(0)}_{11^\star} & U^{(0)}_{11} & U^{(0)}_{12^\star} & U^{(0)}_{12} \\ U^{(0)}_{1^\star 1^\star} & p^2 + U^{(0)}_{1^\star 1} & U^{(0)}_{1^\star 2^\star} & U^{(0)}_{1^\star 2} \\ U^{(0)}_{21^\star} & U^{(0)}_{21} & p^2 + U^{(0)}_{22^\star} & U^{(0)}_{22} \\ U^{(0)}_{2^\star 1^\star} & U^{(0)}_{2^\star 1} & U^{(0)}_{2^\star 2^\star} & p^2 + U^{(0)}_{2^\star 2} \end{pmatrix} , \tag{A.3}$$

with

$$U_{i^{(\star)}\,j^{[\star]}} = \frac{\delta^2 U}{\delta\phi_i^{(\star)}\,\delta\phi_j^{[\star]}} . \tag{A.4}$$

We have then that

$$
\begin{aligned}
\frac{1}{p^8}\det S_E^{(2)} \;=\; & 1 + \frac{2}{p^2}\left(U^{(0)}_{11^\star} + U^{(0)}_{22^\star}\right) \\
& + \frac{1}{p^4}\left((U^{(0)}_{11^\star})^2 + (U^{(0)}_{22^\star})^2 + 4U^{(0)}_{11^\star}U^{(0)}_{22^\star} - U^{(0)}_{11}U^{(0)}_{1^\star 1^\star} - U^{(0)}_{22}U^{(0)}_{2^\star 2^\star} - 2U^{(0)}_{12}U^{(0)}_{1^\star 2^\star} - 2U^{(0)}_{12^\star}U^{(0)}_{1^\star 2}\right) \\
& + \mathscr{O}\left(\frac{1}{p^6}\right) \,,
\end{aligned}
\tag{A.5}
$$

such that, up to finite terms,

$$
\begin{aligned}
\frac{1}{2V^{(4)}}\,\mathrm{STr}\ln S_E^{(2)} \;=\; & \frac{1}{8\pi^2}\int \mathrm{d}p\; p\left(U^{(0)}_{11^\star} + U^{(0)}_{22^\star}\right) \\
& - \int \frac{\mathrm{d}p}{16\pi^2 p}\left((U^{(0)}_{11^\star})^2 + (U^{(0)}_{22^\star})^2 + U^{(0)}_{11}U^{(0)}_{1^\star 1^\star} + U^{(0)}_{22}U^{(0)}_{2^\star 2^\star} + 2U^{(0)}_{12}U^{(0)}_{1^\star 2^\star} + 2U^{(0)}_{12^\star}U^{(0)}_{1^\star 2}\right) ,
\end{aligned}
\tag{A.6}
$$

and substituting the potential (A.1) into this expression gives the one-loop corrections (5.24).

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